On the Probability Measure for Sprinkling in Causal Set Theory

Christoph Minz[1]

[1]Department of Mathematics
University of York

Graduate Research Symposium, 7 April 2019

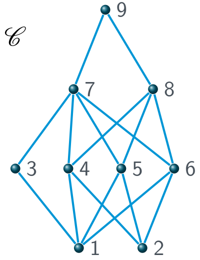
- approach to quantum gravity [Hen09, Sor11]
- replaces continuous spacetime manifold with a discrete set of spacetime events and a causal relation (partial ordered set), $(\mathscr{C}, \preceq)$

$$\begin{aligned} \text{Transitivity:} \quad & x \preceq z \preceq y \implies x \preceq y, \\ \text{Anti-symmetry:} \quad & (x \preceq y \wedge y \preceq x) \implies x = y, \\ \text{Local finiteness:} \quad & |I(x, y)| < \infty, \end{aligned}$$

where the causal interval is

$$I(x, y) := \{z \in \mathscr{C} | x \preceq z \preceq y\}.$$

The relation $\prec$ excludes the case of elements being equal.



Hasse diagram: shows causal relations for the element pairs with $I(x, y) = 2$ ("direct links"). The labelling by integers is a total order preserving the partial order.

	1	2	3	4	5	6	7	8	9
1	0	0	1	1	1	1	0	0	0
2	0	0	0	1	1	1	0	0	0
3	0	0	0	0	0	0	1	0	0
4	0	0	0	0	0	0	1	1	0
5	0	0	0	0	0	0	1	1	0
6	0	0	0	0	0	0	1	1	0
7	0	0	0	0	0	0	0	0	1
8	0	0	0	0	0	0	0	0	1
9	0	0	0	0	0	0	0	0	0

Link matrix: binary representation of the causet links.

Sprinkling is the process of obtaining a causal set (*causet*) from a spacetime:

1. start with a smooth spacetime manifold M (or a compact subset $C \subset M$) and a metric g
2. select sprinkling set $S \underset{\text{finite}}{\subset} M$ by a Poisson process
3. obtain the causal relation from the spacetime metric

$$x \preceq y \Leftrightarrow (J^+(x) \cap J^-(y) \cap C) \text{ is connected}$$

4. forget the embedding of the events in the spacetime, and keep the partial order set $(\mathscr{C}, \preceq)$

1 Introduction
- What is Causal Set Theory?
- Visualisation and Representation of a Causal Set
- What is Sprinkling?

2 Construction of the Probability Measures
- Sprinklings on Compact Spacetime Subsets
- Causet Configurations
- Causet Probability
- Sprinklings on a Spacetime

3 Examples of Small Causets
- 3-Causets in a Diamond Shape
- 3-Causets in a Square And a Circle Shape
- 4-Causets

4 Future and Past in Causal Sets
- Smallest Timelike Distances in Causal Sets
- Propagators of Quantum Fields on Causal Sets

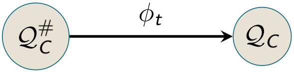
- C: compact subset of the spacetime manifold M
- configuration space of ordered sprinklings (distinguishable events)

$$\mathcal{Q}_C^{\#} := \bigoplus_{N=0}^{\infty} C^N$$

- configuration space of sprinklings (indistinguishable events) using the permutation group P_N

$$\mathcal{Q}_C := \bigoplus_{N=0}^{\infty} C^N / P_N$$

- homomorphism removing the label due to time-ordering



- alternative the labels might follow from the light-cone coordinate $u = \frac{1}{\sqrt{2}}(t + r)$, where r is the Euclidean distance for spacelike coordinates

- volume-density parameter ρ, determines how many events are sprinkled in the volume

$$V(C) = \int_C \mathrm{dvol}_i$$

- probability measure $\mu^{\#}_{C,\rho} : \mathcal{Q}^{\#}_C \to \mathbb{R}^+$ for ordered sprinklings such that

$$\mathrm{d}\mu^{\#}_{C,\rho} = \frac{1}{a} \sum_{N=0}^{\infty} \rho^N \prod_{i=1}^{N} \mathrm{dvol}_i$$

- push-forward of the probability measure $\mu_{C,\rho} = \phi_{t*}\mu^{\#}_{C,\rho}$

$$\mathrm{d}\mu_{C,\rho} = \frac{1}{a} \sum_{N=0}^{\infty} \rho^N \prod_{i=2}^{N} \Theta(t_i - t_{i-1}) \prod_{i=1}^{N} \mathrm{dvol}_i, \qquad a = \mathrm{e}^{\rho V(C)}$$

$$= \sum_{N=0}^{\infty} \underbrace{\frac{1}{N!} \left(\rho V(C)\right)^N \mathrm{e}^{-\rho V(C)}}_{\Pr(|\cdot|=N)} \frac{N!}{V(C)^N} \prod_{i=2}^{N} \Theta(t_i - t_{i-1}) \prod_{i=1}^{N} \mathrm{dvol}_i$$

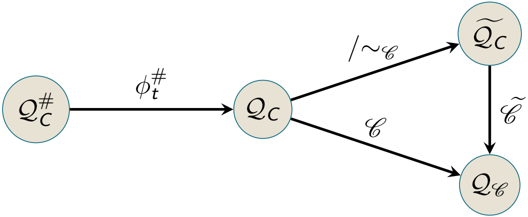
- configuration space of causets

$$\mathcal{Q}_{\mathscr{C}} = \bigoplus_{N=0}^{\infty} \text{Posets}(N)$$

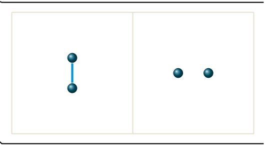
- homomorphism from sprinklings to causets $\mathscr{C} : \mathcal{Q}_C \to \mathcal{Q}_{\mathscr{C}}$
- causet equivalence and configuration space quotient

$$S \sim_{\mathscr{C}} S' \Leftrightarrow \mathscr{C}(S) = \mathscr{C}(S'), \qquad \widetilde{\mathcal{Q}_C} = \mathcal{Q}_C / \sim_{\mathscr{C}}$$

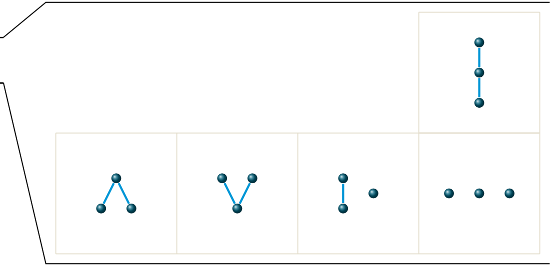
- homomorphism from sprinklings to causets



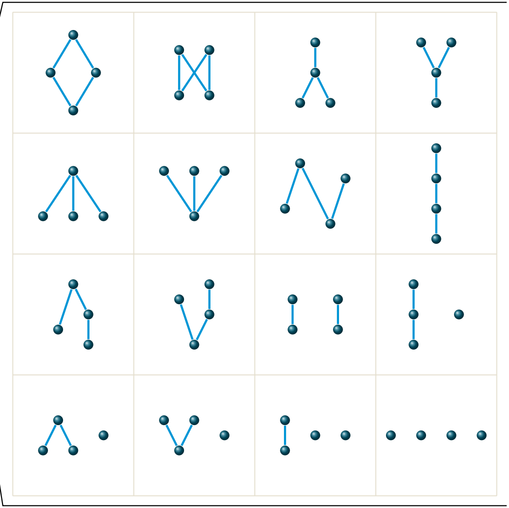
N	$a_{A000112}(N)$
0	1
1	1
2	**2**
3	5
4	16
5	63
6	318
7	2045
8	16999
9	183231
10	2567284
11	46749427
12	1104891746



N	$a_{A000112}(N)$
0	1
1	1
2	2
3	**5**
4	16
5	63
6	318
7	2045
8	16999
9	183231
10	2567284
11	46749427
12	1104891746



N	$a_{A000112}(N)$
0	1
1	1
2	2
3	5
4	**16**
5	63
6	318
7	2045
8	16999
9	183231
10	2567284
11	46749427
12	1104891746



The probability for one equivalence class $[S]_{\mathscr{C}}$, which yields the same causet $\mathscr{C}(S)$ with a fixed number N of elements is
$P = \Pr\big([S]_{\mathscr{C}} \;\big|\; |S| = N\big)$,

$$P = \frac{N!}{V(C)^N} \int_{\mathcal{Q}_C} \Theta\Big((x_1, \dots, x_N) \in [S]_C\Big) \prod_{i=2}^{N} \Theta(t_i - t_{i-1}) \prod_{i=1}^{N} \mathrm{dvol}_i$$

$$= \frac{N!}{V(C)^N} \int_{\mathcal{Q}_C} \prod_{i=2}^{N} \Theta\Big(x_i \in C_i^{\mathscr{C}}\Big) \prod_{i=2}^{N} \Theta(t_i - t_{i-1}) \prod_{i=1}^{N} \mathrm{dvol}_i$$

$$C_i^{\mathscr{C}} = C \cap \bigcap_{j=1}^{i-1} \begin{cases} J^+(x_j), & \mathscr{C}_j(S) \preceq \mathscr{C}_i(S) \\ C \setminus J^+(x_j), & \text{else} \end{cases}$$

- all compact subsets of a spacetime $L := \left\{ C \underset{\text{compact}}{\subset} M \right\}$
- configuration spaces are related by projectors such that $\forall C_1, C_2, C_3 \in L$

$$C_2 \supseteq C_1 \implies \Pi_{C_1 C_2} : \mathcal{Q}_{C_2} \to \mathcal{Q}_{C_1}$$
$$C_3 \supseteq C_2 \supseteq C_1 \implies \Pi_{C_1 C_2} \Pi_{C_2 C_3} = \Pi_{C_1 C_3}$$

- projective limit of configuration spaces

$$\overleftarrow{\mathcal{Q}} = \varprojlim_{C \in L} \mathcal{Q}_C := \left\{ (S_C)_{C \in L} \in \prod_{C \in L} \mathcal{Q}_C \,\middle|\, \forall C_2 \supseteq C_1 \in L : \Pi_{C_1 C_2} S_2 = S_1 \right\}$$

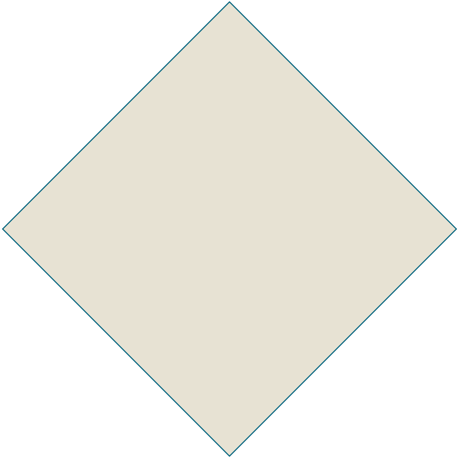
- natural projectors on the limit

$$\Pi_C : \overleftarrow{\mathcal{Q}} \to \mathcal{Q}_C, \qquad (S_{C'})_{C' \in L} \mapsto S_C$$

- probability measure [AL95] such that

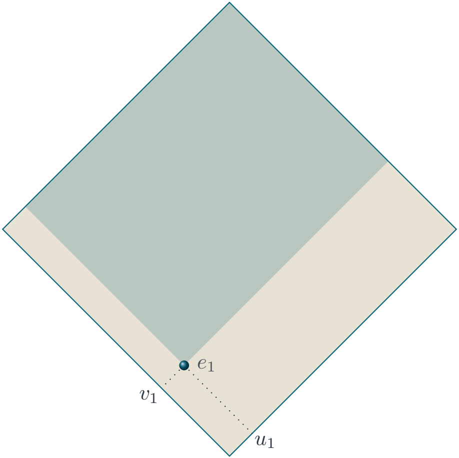
$$\Pi_{C*} \mu_\rho = \mu_{C,\rho}, \qquad \int_{\mathcal{Q}_C} f \mathrm{d}\mu_{C,\rho} = \int_{\overleftarrow{\mathcal{Q}}} f \circ \Pi_C \mathrm{d}\mu_\rho$$

Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ all ordered



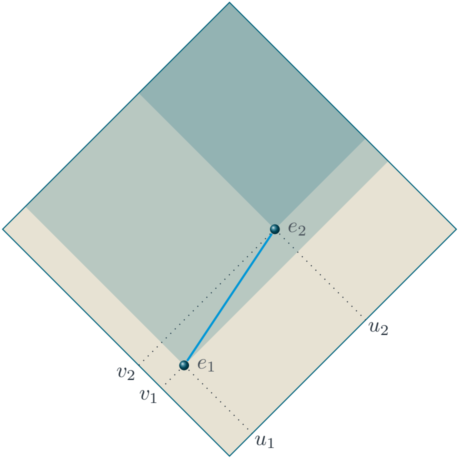
$$\Pr(\Sigma_{123}) = \int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}v_1 \int_{u_1}^1 \mathrm{d}u_2 \int_{v_1}^1 \mathrm{d}v_2 \int_{u_2}^1 \mathrm{d}u_3 \int_{v_2}^1 \mathrm{d}v_3$$

Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ all ordered



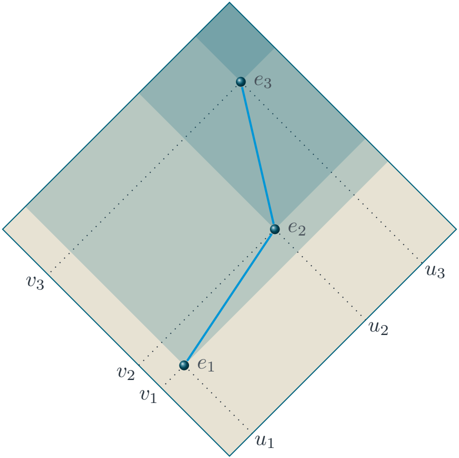
$$\Pr(\Sigma_{123}) = 1! \int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}v_1 \int_{u_1}^1 \mathrm{d}u_2 \int_{v_1}^1 \mathrm{d}v_2 \int_{u_2}^1 \mathrm{d}u_3 \int_{v_2}^1 \mathrm{d}v_3 = 1$$

Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ all ordered



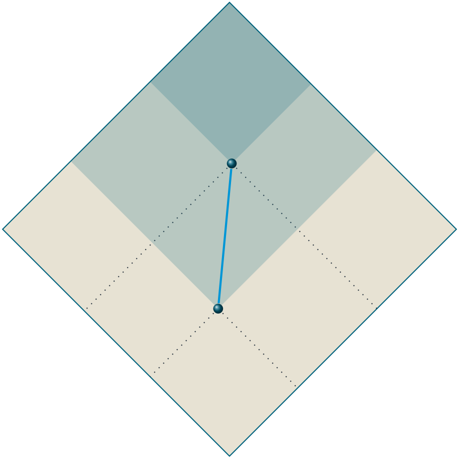
$$\Pr(\Sigma_{123}) = 2! \int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}v_1 \int_{u_1}^1 \mathrm{d}u_2 \int_{v_1}^1 \mathrm{d}v_2 \int_{u_2}^1 \mathrm{d}u_3 \int_{v_2}^1 \mathrm{d}v_3 = \frac{1}{2!}$$

Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ all ordered

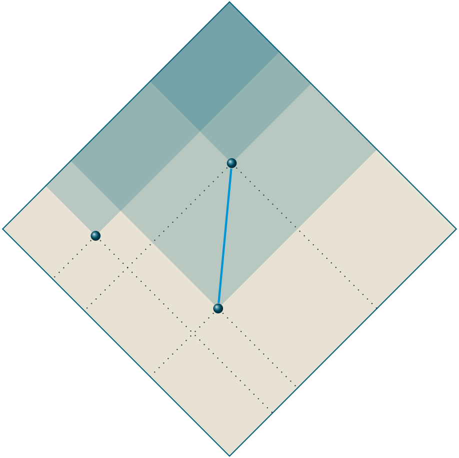


$$\Pr(\Sigma_{123}) = 3! \int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}v_1 \int_{u_1}^1 \mathrm{d}u_2 \int_{v_1}^1 \mathrm{d}v_2 \int_{u_2}^1 \mathrm{d}u_3 \int_{v_2}^1 \mathrm{d}v_3 = \frac{1}{3!}$$

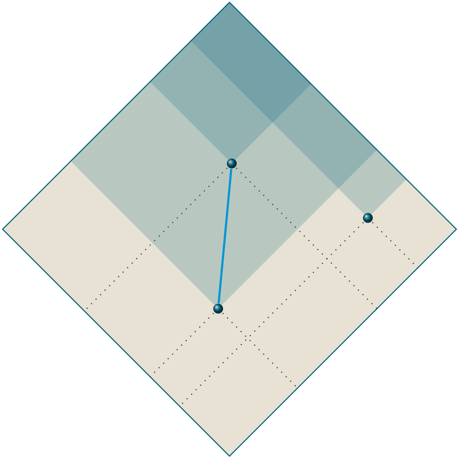
Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered



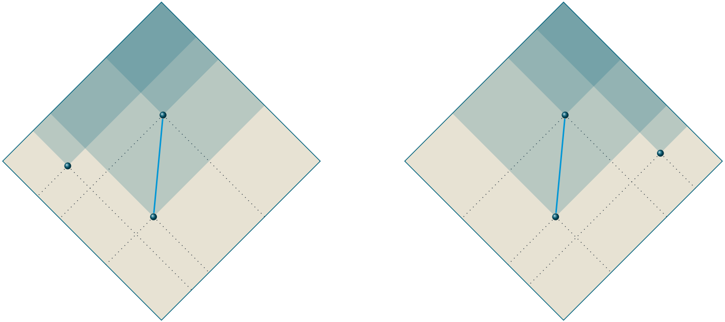
Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered



Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered

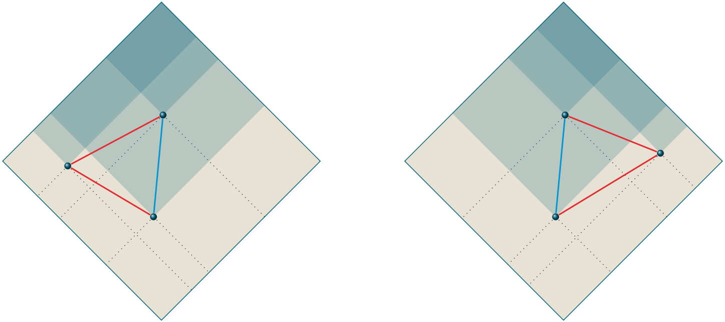


Example: $d = 1+1$, diamond shape (unit volume), $N = 3$ two ordered



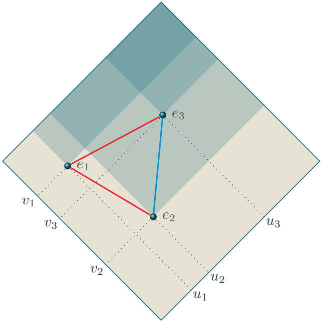
$\Pr(\Sigma_{231}) = \frac{1}{3!}$ $\Pr(\Sigma_{312}) = \frac{1}{3!}$

Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered

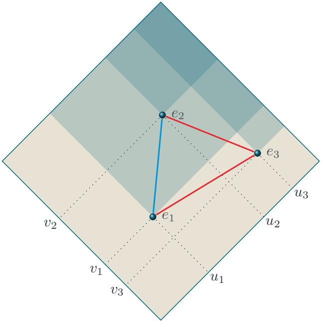


$\Pr(\Sigma_{231}) = \frac{1}{3!}$ $\Pr(\Sigma_{312}) = \frac{1}{3!}$

Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered

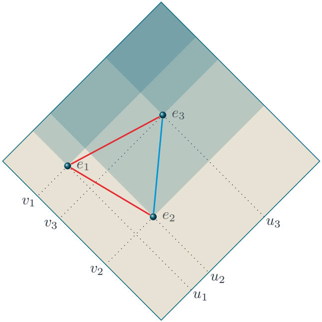


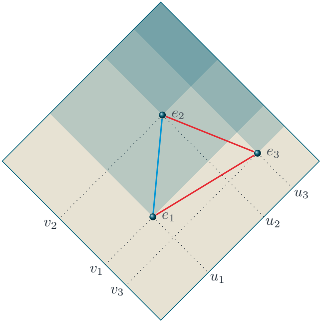
$$\Pr(\Sigma_{231}) = \frac{1}{3!}$$



$$\Pr(\Sigma_{312}) = \frac{1}{3!}$$

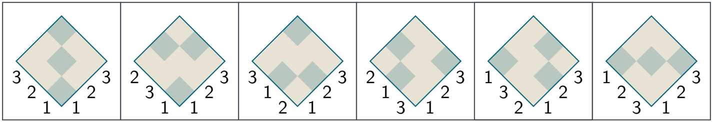
Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered



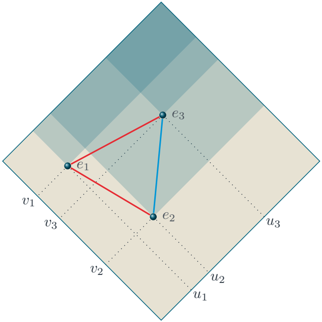


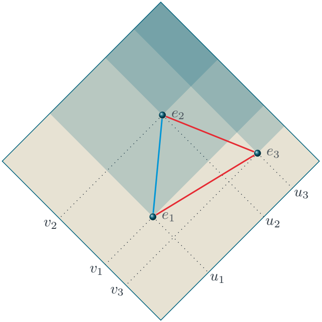
$$\Pr(\Sigma_{231}) = \frac{1}{3!}$$

$$\Pr(\Sigma_{312}) = \frac{1}{3!}$$

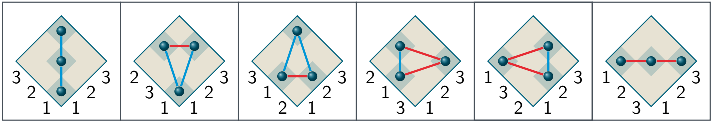


Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered

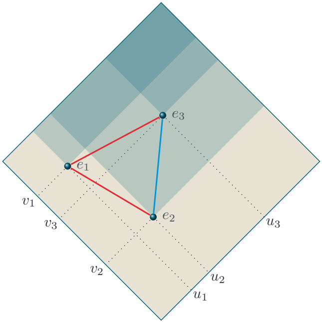




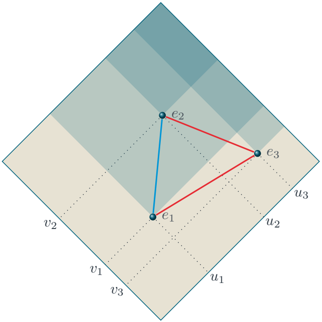
$$\Pr(\Sigma_{231}) = \frac{1}{3!} \qquad \Pr(\Sigma_{312}) = \frac{1}{3!}$$



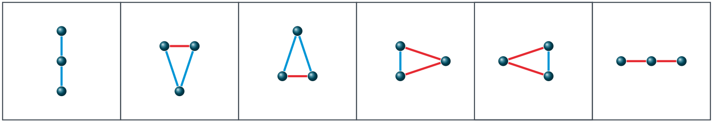
Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered



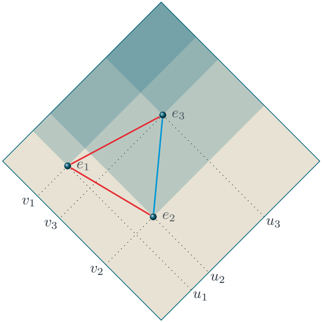
$$\Pr(\Sigma_{231}) = \frac{1}{3!}$$

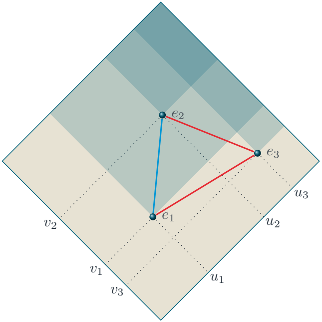


$$\Pr(\Sigma_{312}) = \frac{1}{3!}$$



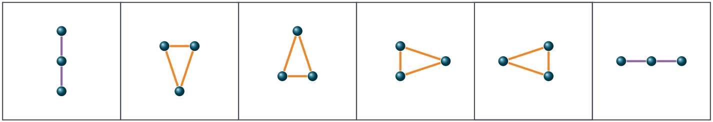
Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered



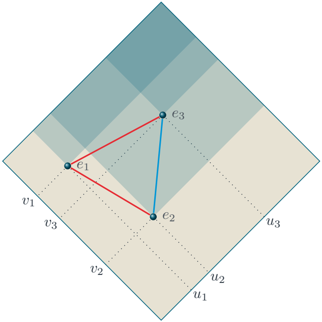


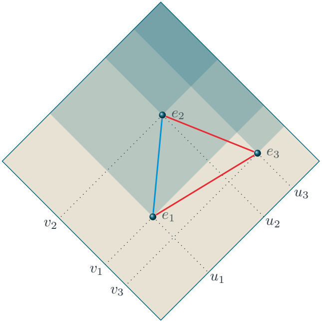
$$\Pr(\Sigma_{231}) = \frac{1}{3!}$$

$$\Pr(\Sigma_{312}) = \frac{1}{3!}$$



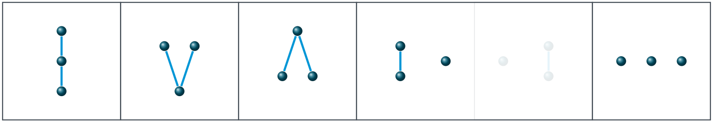
Example: $d = 1 + 1$, diamond shape (unit volume), $N = 3$ two ordered



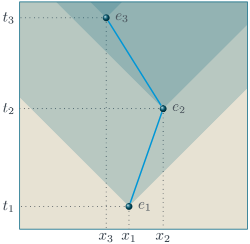


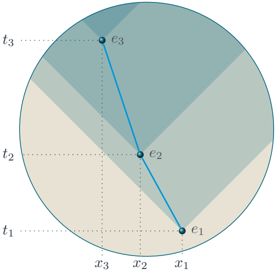
$$\Pr(\Sigma_{231}) = \frac{1}{3!}$$

$$\Pr(\Sigma_{312}) = \frac{1}{3!}$$



Example: $d = 1+1$, other unit volumes, $N = 3$ all ordered

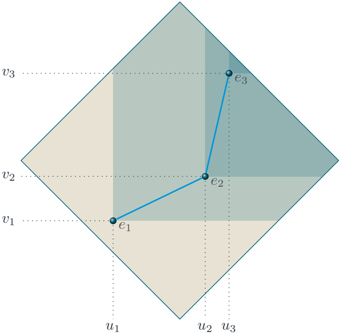


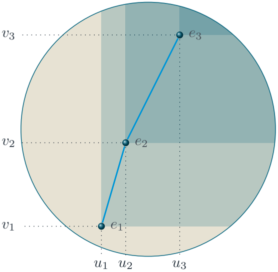


$$\Pr{}_{\square}(\Sigma_{123}) = 3! \int_{-a}^{a} \mathrm{d}u_1 \int_{-a+|u_1|}^{a-|u_1|} \mathrm{d}v_1 \int_{u_1}^{a-|v_1|} \mathrm{d}u_2 \int_{\max(v_1,-a+u_2)}^{a-|u_2|} \mathrm{d}v_2 \int_{u_2}^{a-|v_2|} \mathrm{d}u_3 \int_{\max(v_2,-a+u_3)}^{a-|u_3|} \mathrm{d}v_3$$

$$\Pr{}_{\bigcirc}(\Sigma_{123}) = 3! \int_{-a}^{a} \mathrm{d}u_1 \int_{-\sqrt{a^2-u_1^2}}^{\sqrt{a^2-u_1^2}} \mathrm{d}v_1 \int_{u_1}^{\sqrt{a^2-v_1^2}} \mathrm{d}u_2 \int_{\max(v_1,-\sqrt{a^2-u_2^2})}^{\sqrt{a^2-u_2^2}} \mathrm{d}v_2 \int_{u_2}^{\sqrt{a^2-v_2^2}} \mathrm{d}u_3 \int_{\max(v_1,-\sqrt{a^2-u_3^2})}^{\sqrt{a^2-u_3^2}} \mathrm{d}v_3$$

Example: $d = 1+1$, other unit volumes, $N = 3$ all ordered

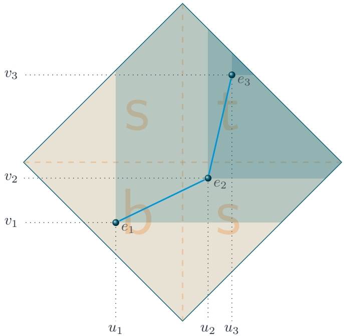


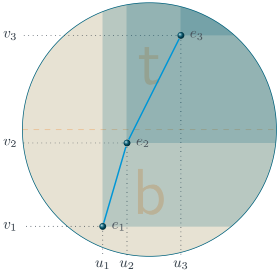


$$\Pr{}_{\square}(\Sigma_{123}) = 3! \int_{-a}^{a} \mathrm{d}u_1 \int_{-a+|u_1|}^{a-|u_1|} \mathrm{d}v_1 \int_{u_1}^{a-|v_1|} \mathrm{d}u_2 \int_{\max(v_1,-a+u_2)}^{a-|u_2|} \mathrm{d}v_2 \int_{u_2}^{a-|v_2|} \mathrm{d}u_3 \int_{\max(v_2,-a+u_3)}^{a-|u_3|} \mathrm{d}v_3$$

$$\Pr{}_{\bigcirc}(\Sigma_{123}) = 3! \int_{-a}^{a} \mathrm{d}u_1 \int_{-\sqrt{a^2-u_1^2}}^{\sqrt{a^2-u_1^2}} \mathrm{d}v_1 \int_{u_1}^{\sqrt{a^2-v_1^2}} \mathrm{d}u_2 \int_{\max(v_1,-\sqrt{a^2-u_2^2})}^{\sqrt{a^2-u_2^2}} \mathrm{d}v_2 \int_{u_2}^{\sqrt{a^2-v_2^2}} \mathrm{d}u_3 \int_{\max(v_1,-\sqrt{a^2-u_3^2})}^{\sqrt{a^2-u_3^2}} \mathrm{d}v_3$$

Example: $d = 1 + 1$, other unit volumes, $N = 3$ all ordered

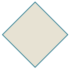
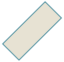
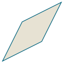
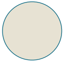
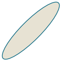




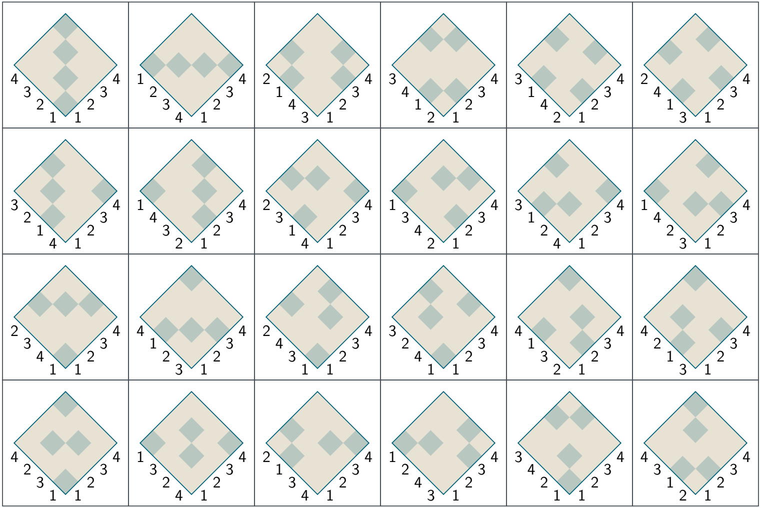
$$\mathrm{Pr}_{\square}(\Sigma_{123}) = 3!\,\left(2\int_{\mathrm{ttt}} + 4\int_{\mathrm{stt}} + 4\int_{\mathrm{sst}} + 2\int_{\mathrm{sss}} + 2\int_{\mathrm{btt}} + 2\int_{\mathrm{bst}}\right) = \frac{17}{120}$$

$$\mathrm{Pr}_{\bigcirc}(\Sigma_{123}) = 3!\,\left(2\int_{\mathrm{ttt}} + 2\int_{\mathrm{btt}}\right) = \frac{1}{8} + \frac{1}{4\pi^2}$$

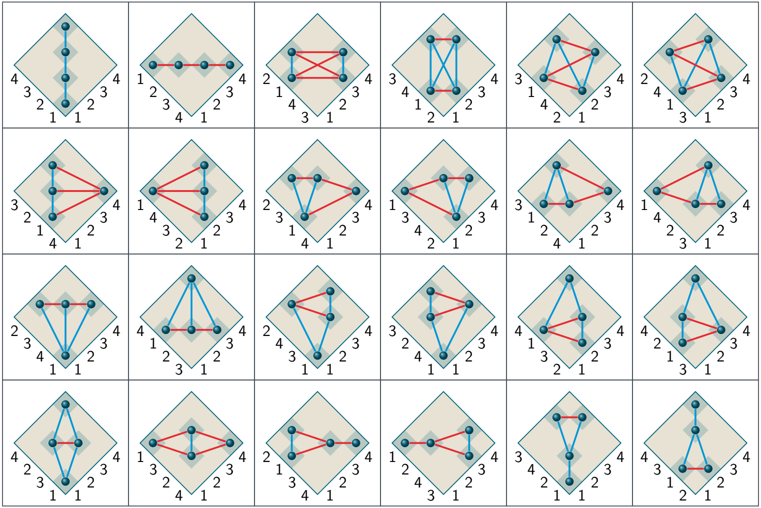
3-Causets Summary

compact vol. of Minkowski spacetime / causets	 diamond	 boosted diamond	 square	 boosted square	 circle	 boosted circle
 123	0.1668 0.1667 1	0.1666	0.1418 0.1417 34	0.1415	0.1505 0.1503 $2\pi^2+4$	0.1502
 132	0.1664 0.1667 1	0.1665	0.1789 0.1792 43	0.1795	0.1749 0.1748 $3\pi^2-2$	0.1750
 213	0.1668 0.1667 1	0.1669	0.1791 0.1792 43	0.1793	0.1745 0.1748 $3\pi^2-2$	0.1751
 231, 312	0.3335 0.3333 2	0.3333	0.3586 0.3583 86	0.3581	0.3496 0.3497 $6\pi^2-4$	0.3494
 321	0.1665 0.1667 1	0.1668	0.1416 0.1417 34	0.1416	0.1505 0.1503 $2\pi^2+4$	0.1504
ratio sum.	6		240		$16\pi^2$	

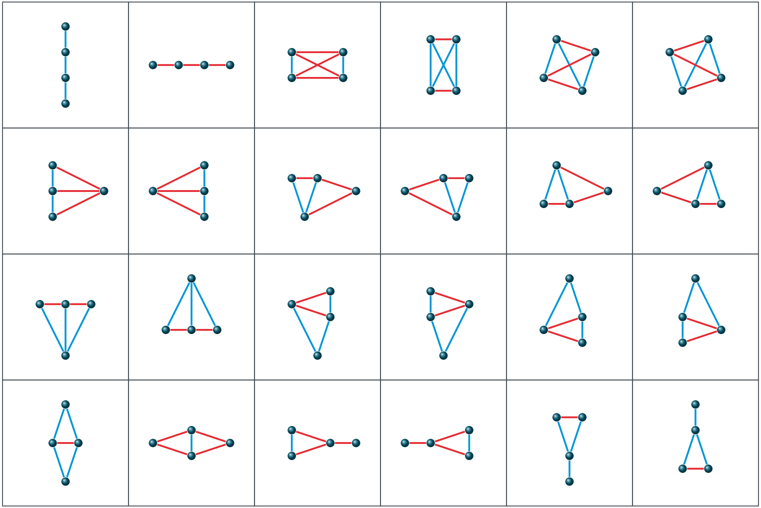
Example: $d = 1 + 1$, $N = 4$, permutations → permutations and bi-posets → bi-posets → bi-poset groups → poset (probabilities)



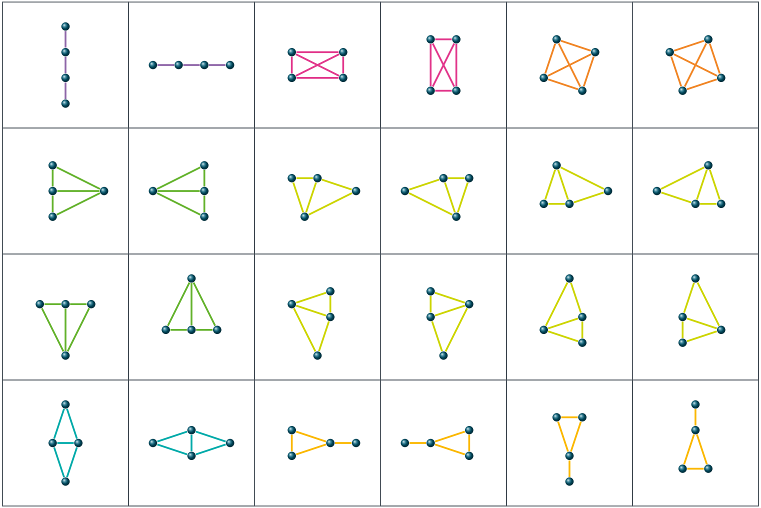
Example: $d = 1 + 1$, $N = 4$, permutations $\rightarrow$ permutations and bi-posets $\rightarrow$ bi-posets $\rightarrow$ bi-poset groups $\rightarrow$ poset (probabilities)



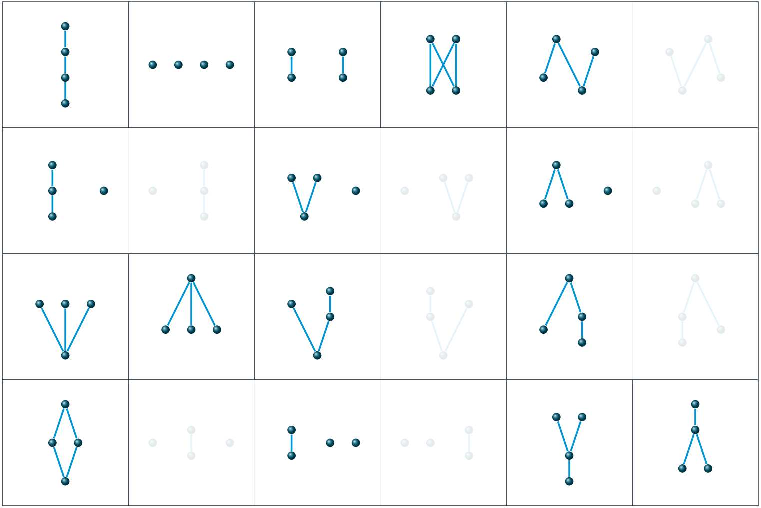
Example: $d = 1+1$, $N = 4$, permutations $\rightarrow$ permutations and bi-posets $\rightarrow$ bi-posets $\rightarrow$ bi-poset groups $\rightarrow$ poset (probabilities)



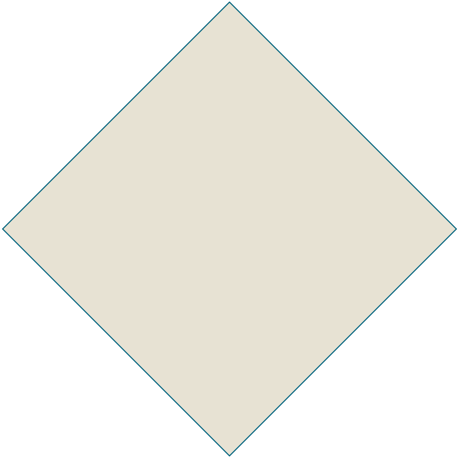
Example: $d = 1 + 1$, $N = 4$, permutations $\to$ permutations and bi-posets $\to$ bi-posets $\to$ bi-poset groups $\to$ poset (probabilities)



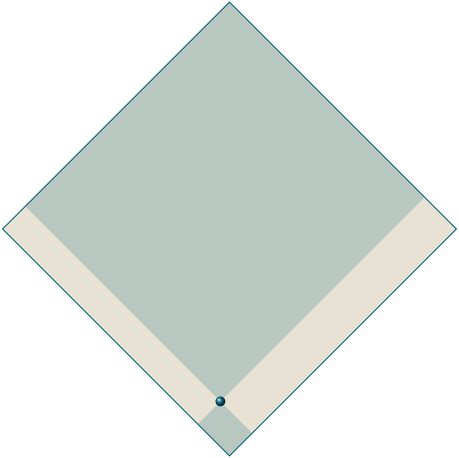
Example: $d = 1 + 1$, $N = 4$, permutations $\to$ permutations and bi-posets $\to$ bi-posets $\to$ bi-poset groups $\to$ poset (probabilities)



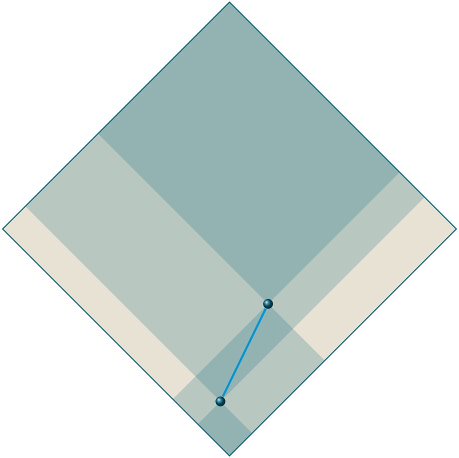
Sprinkling in a $1 + 1$ dimensional spacetime volume to find diamonds



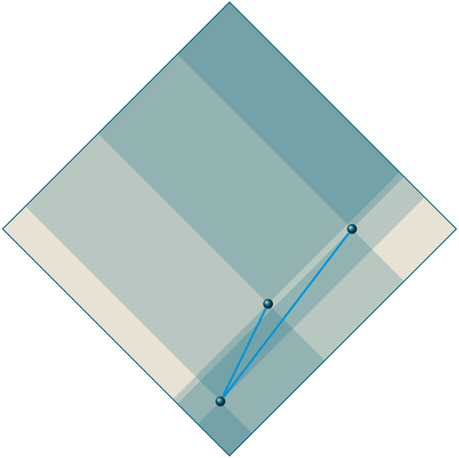
Sprinkling in a $1+1$ dimensional spacetime volume to find diamonds



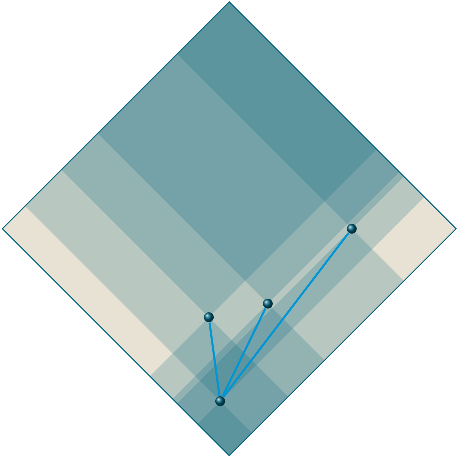
Sprinkling in a $1+1$ dimensional spacetime volume to find diamonds



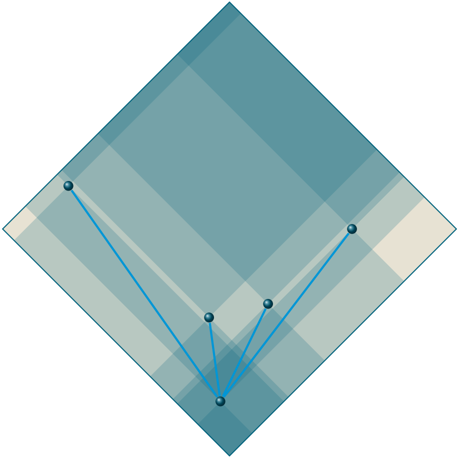
Sprinkling in a $1+1$ dimensional spacetime volume to find diamonds



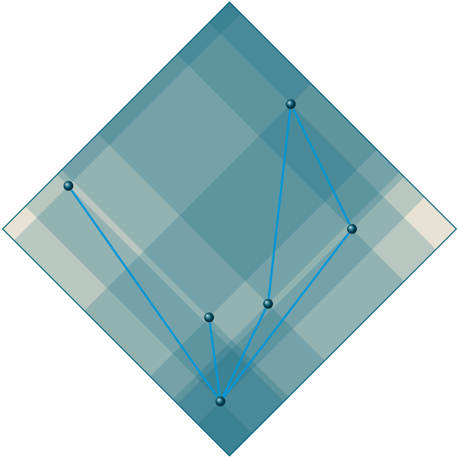
Sprinkling in a $1 + 1$ dimensional spacetime volume to find diamonds



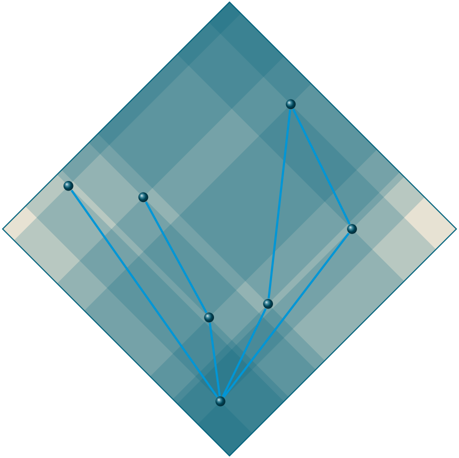
Sprinkling in a $1+1$ dimensional spacetime volume to find diamonds



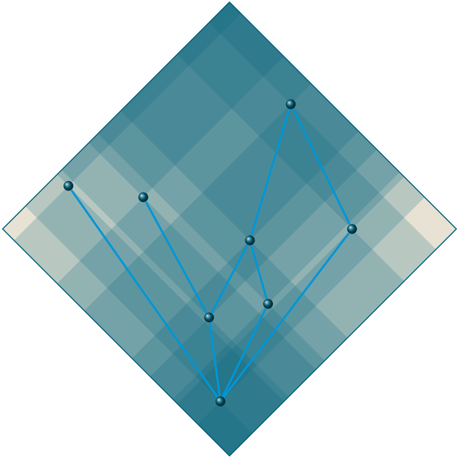
Sprinkling in a $1+1$ dimensional spacetime volume to find diamonds



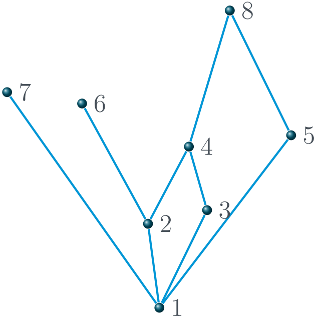
Sprinkling in a $1 + 1$ dimensional spacetime volume to find diamonds



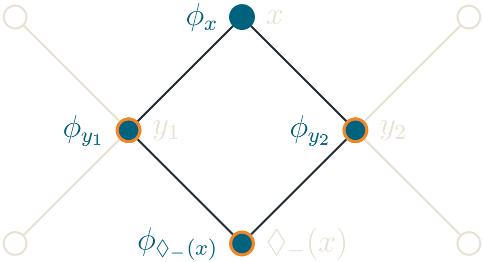
Sprinkling in a $1+1$ dimensional spacetime volume to find diamonds



Diamond-links as a relation for timelike distance in causal sets



	1	2	3	4	5	6	7	8
1	0	1	1	0	1	0	1	0
2	0	0	0	1	0	1	0	0
3	0	0	0	1	0	0	0	0
4	0	0	0	0	0	0	0	1
5	0	0	0	0	0	0	0	1
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0



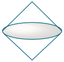
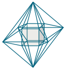
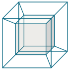
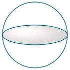
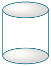
$$(\Box_\pm \phi)_x := \rho^{\frac{2}{d}} \left(\phi_x + \frac{1}{|\Diamond_\pm(x)|} \sum_{z \in \Diamond_\pm(x)} \left(\phi_z - \frac{\alpha}{|\blacklozenge_\pm(x,z)|} \sum_{y \in \blacklozenge_\pm(x,z)} \phi_y \right) \right)$$

Abhay Ashtekar and Jerzy Lewandowski.
Projective Techniques and Functional Integration for Gauge Theories.
Journal of Mathematical Physics, 36(5):2170–2191, 1995.

Joe Henson.
The Causal Set Approach to Quantum Gravity.
Approaches to Quantum Gravity: Towards a New Understanding of Space, Time and Matter, 393, 2009.

Rafael D Sorkin.
Scalar Field Theory on a Causal Set in Histories Form.
In *Journal of Physics: Conference Series*, volume 306, 2011.

compact vol. of Minkowski spacetime / causets	 double cone	 boosted double cone	 octahedron (diamond)	 cube	 ball	 cylinder
 123	0.0229 0.0256 1	0.0228	0.0425 0.0435 7	0.0245 0.0248 3	0.0305 0.0299 4	0.0371 0.0376 5
 132	0.0773 0.0769 3	0.0770	0.1123 0.1118 18	0.1082 0.1074 13	0.1106 0.1119 15	0.1426 0.1429 19
 213	0.0774 0.0769 3	0.0773	0.1122 0.1118 18	0.1079 0.1074 13	0.1107 0.1119 15	0.1424 0.1429 19
 231, 312	0.3087 0.3077 12	0.3086	0.3158 0.3168 51	0.3646 0.3636 44	0.3449 0.3433 46	0.3519 0.3534 47
 321	0.5138 0.5128 20	0.5143	0.4173 0.4161 67	0.3948 0.3967 48	0.4033 0.4030 54	0.3261 0.3233 43
ratio sum.	39		161	121	134	133

compact vol. of Minkowski spacetime / causets	 double cone	 diamond	 tesseract (cube)	 ball	 cylinder
 123	0.0029 0.0053 1	0.0109 0.0092 1	0.0042 0.0060 1	0.0069 0.0074 1	0.0127 0.0125 1
 132	0.0258 0.0266 5	0.0614 0.0642 7	0.0534 0.0542 9	0.0610 0.0593 8	0.1121 0.1125 9
 213	0.0258 0.0266 5	0.0614 0.0642 7	0.0536 0.0542 9	0.0610 0.0593 8	0.1126 0.1125 9
 231, 312	0.1882 0.1862 35	0.2493 0.2477 27	0.2976 0.2952 49	0.2808 0.2815 38	0.3262 0.3250 26
 321	0.7574 0.7553 142	0.6171 0.6147 67	0.5911 0.5904 98	0.5904 0.5926 80	0.4364 0.4375 35
ratio sum.	188	109	166	135	80

4-Causets Summary

causets \ compact vol. of Minkowski spacetime	diamond	square	circle
1234	0.0416 0.0417 1	0.0270 0.0282 7	0.0315 0.0309 15
4321	0.0417 0.0417 1	0.0269 0.0282 7	0.0313 0.0309 15
2143	0.0417 0.0417 1	0.0619 0.0605 15	0.0537 0.0535 26
3142, 2413	0.0835 0.0833 2	0.1130 0.1129 28	0.1035 0.1029 50
3412	0.0415 0.0417 1	0.0619 0.0605 15	0.0538 0.0535 26
1243	0.0418 0.0417 1	0.0404 0.0403 10	0.0413 0.0412 20
2134	0.0417 0.0417 1	0.0402 0.0403 10	0.0413 0.0412 20
3421, 4312, 4231	0.1248 0.1250 3	0.1046 0.1048 26	0.1130 0.1132 55
2341, 4123	0.0837 0.0833 2	0.0887 0.0887 22	0.0852 0.0864 42
2431, 4132	0.0835 0.0833 2	0.0807 0.0806 20	0.0826 0.0823 40
3241, 4213	0.0829 0.0833 2	0.0806 0.0806 20	0.0820 0.0823 40
1423, 1342	0.0834 0.0833 2	0.0805 0.0806 20	0.0827 0.0823 40
3124, 2314	0.0834 0.0833 2	0.0807 0.0806 20	0.0824 0.0823 40
1432	0.0417 0.0417 1	0.0443 0.0444 11	0.0426 0.0432 21
3214	0.0415 0.0417 1	0.0444 0.0444 11	0.0427 0.0432 21
1324	0.0417 0.0417 1	0.0242 0.0242 6	0.0305 0.0309 15
ratio sum.	24	248	486

causets \ compact vol. of Minkowski spacetime	double cone	octahedron (diamond)	cube	ball	cylinder
1234	0.0013 0.0017 1	0.0035 0.0031 2	0.0009 0.0015 1	0.0015 0.0015 1	0.0017 0.0015 1
4321	0.3091 0.3090 186	0.2198 0.2202 144	0.1819 0.1818 120	0.1973 0.1975 129	0.1322 0.1319 86
2143	0.0151 0.0150 9	0.0253 0.0260 17	0.0327 0.0333 22	0.0309 0.0306 20	0.0522 0.0521 34
3142, 2413	0.0449 0.0449 27	0.0614 0.0612 40	0.0831 0.0833 55	0.0776 0.0781 51	0.1039 0.1043 68
3412	0.0172 0.0166 10	0.0187 0.0183 12	0.0377 0.0379 25	0.0277 0.0276 18	0.0341 0.0337 22
1243	0.0043 0.0050 3	0.0093 0.0092 6	0.0051 0.0045 3	0.0067 0.0061 4	0.0093 0.0092 6
2134	0.0043 0.0050 3	0.0093 0.0092 6	0.0051 0.0045 3	0.0067 0.0061 4	0.0092 0.0092 6
3421, 4312, 4231	0.2825 0.2824 170	0.2429 0.2431 159	0.2688 0.2682 177	0.2585 0.2588 169	0.2173 0.2178 142
2341, 4123	0.0259 0.0266 16	0.0316 0.0321 21	0.0302 0.0303 20	0.0300 0.0306 20	0.0332 0.0337 22
2431, 4132	0.0994 0.0997 60	0.1055 0.1055 69	0.1187 0.1182 78	0.1137 0.1133 74	0.1167 0.1166 76
3241, 4213	0.0995 0.0997 60	0.1052 0.1055 69	0.1187 0.1182 78	0.1136 0.1133 74	0.1167 0.1166 76
1423, 1342	0.0172 0.0166 10	0.0322 0.0321 21	0.0193 0.0197 13	0.0248 0.0245 16	0.0311 0.0307 20
3124, 2314	0.0172 0.0166 10	0.0324 0.0321 21	0.0193 0.0197 13	0.0247 0.0245 16	0.0311 0.0307 20
1432	0.0287 0.0282 17	0.0457 0.0459 30	0.0380 0.0379 25	0.0407 0.0413 27	0.0531 0.0537 35
3214	0.0288 0.0282 17	0.0459 0.0459 30	0.0380 0.0379 25	0.0407 0.0413 27	0.0533 0.0537 35
1324	0.0043 0.0050 3	0.0113 0.0107 7	0.0025 0.0030 2	0.0049 0.0046 3	0.0048 0.0046 3
ratio sum.	602	654	660	653	652

causets \ compact vol. of Minkowski spacetime	double cone	diamond	tesseract (cube)	ball	cylinder
1234	0.0000 0.0001 1	0.0003 0.0006 1	0.0000 0.0000 1	0.0001 0.0001 1	0.0002 0.0003 1
4321	0.6063 0.6062 9077	0.4326 0.4320 680	0.3782 0.3782 8442	0.3887 0.3886 2692	0.2268 0.2266 732
2143	0.0030 0.0030 45	0.0100 0.0102 16	0.0110 0.0110 246	0.0127 0.0127 88	0.0386 0.0384 124
3142, 2413	0.0128 0.0128 192	0.0319 0.0318 50	0.0399 0.0399 890	0.0422 0.0421 292	0.0848 0.0848 274
3412	0.0048 0.0048 72	0.0079 0.0076 12	0.0208 0.0208 464	0.0135 0.0136 94	0.0212 0.0214 69
1243	0.0003 0.0003 5	0.0017 0.0019 3	0.0005 0.0005 11	0.0010 0.0010 7	0.0024 0.0025 8
2134	0.0003 0.0003 5	0.0017 0.0019 3	0.0005 0.0005 11	0.0010 0.0010 7	0.0025 0.0025 8
3421, 4312, 4231	0.2415 0.2415 3616	0.2581 0.2579 406	0.3153 0.3153 7039	0.2883 0.2883 1997	0.2536 0.2533 818
2341, 4123	0.0050 0.0049 74	0.0108 0.0108 17	0.0076 0.0077 171	0.0091 0.0091 63	0.0136 0.0136 44
2431, 4132	0.0516 0.0516 773	0.0831 0.0832 131	0.0922 0.0922 2059	0.0920 0.0919 637	0.1195 0.1195 386
3241, 4213	0.0516 0.0516 773	0.0832 0.0832 131	0.0922 0.0922 2059	0.0918 0.0919 637	0.1196 0.1195 386
1423, 1342	0.0022 0.0022 33	0.0100 0.0102 16	0.0033 0.0033 74	0.0063 0.0064 44	0.0125 0.0127 41
3124, 2314	0.0022 0.0022 33	0.0100 0.0102 16	0.0033 0.0033 74	0.0063 0.0064 44	0.0127 0.0127 41
1432	0.0090 0.0090 135	0.0280 0.0280 44	0.0174 0.0174 389	0.0231 0.0231 160	0.0454 0.0455 147
3214	0.0090 0.0090 135	0.0281 0.0280 44	0.0174 0.0174 389	0.0231 0.0231 160	0.0456 0.0455 147
1324	0.0003 0.0003 5	0.0025 0.0025 4	0.0002 0.0002 4	0.0007 0.0007 5	0.0011 0.0012 4
ratio sum.	14974	1574	22323	6928	3230