Diamonds and Local Structure in Sprinkled Causal Sets

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Causal set theory is a framework for quantum gravity [2,4], which replaces the classical spacetime continuum by the discrete structure of a causal set at a small length scale. We investigate the Poisson process of *sprinkling* that is commonly used to construct causal sets from a given spacetime manifold.

1. What Is a Causal Set?

A *causal set (or causet)* $(\mathscr{C}, \preceq)$ is a locally finite, partially ordered set. The (finite) *Alexandrov interval* of two events $x \preceq y \in \mathscr{C}$ is defined as

$$I(x, y) := \{z \in \mathscr{C} \mid x \preceq z \preceq y\}. \tag{1}$$

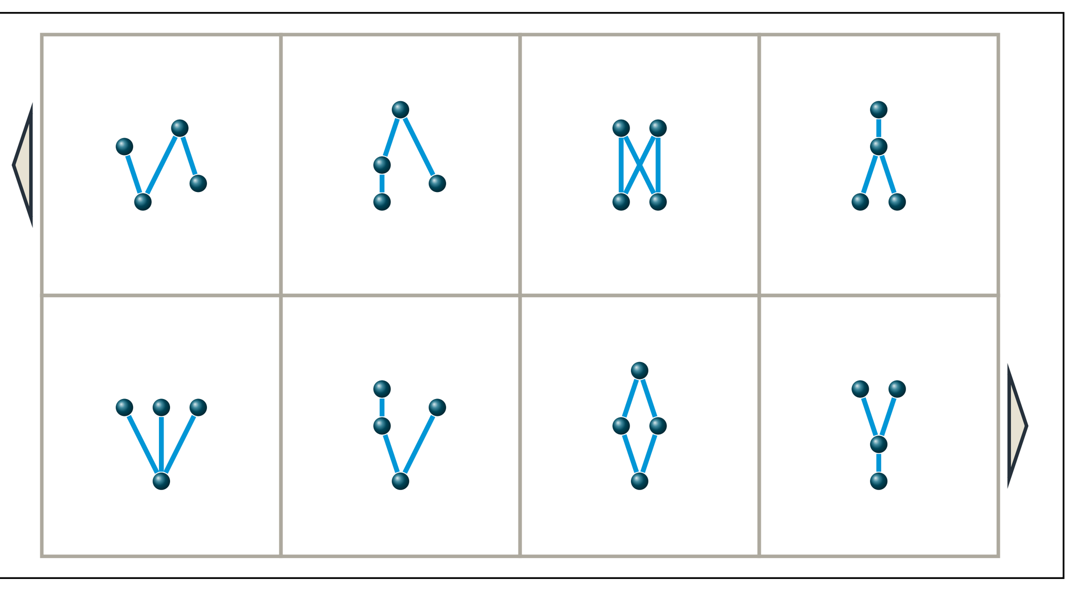
An event $x \in \mathscr{C}$ is *linked to* another event $y \in \mathscr{C}$ when

$$x \prec\!\ast\, y \Leftrightarrow (I(x, y) = \{x, y\} \wedge x \neq y). \tag{2}$$

Events and their links are graphically represented by *Hasse diagrams*.

The number of possible causets $a(n)$ with cardinality $|\mathscr{C}| = n$. We show some causet examples below.

n	$a(n)$
⋮	⋮
2	2
3	5
4	**16**
5	63
6	318
⋮	⋮



2. Poisson Probability Measure

We derive the Poisson probability measure for sprinkling from the intensity measure λ on the manifold such that for every open subset $O \subset M$,

$$\lambda(O) = \rho \int_O \sqrt{|g|}\, \mathrm{d}^d x \tag{3}$$

is the volume of O multiplied by the *sprinkling density* ρ. We can define n-fold measures $\lambda_{U,n}$ on subsets $U \in L$, so that the Poisson probability measure on open, pre-compact subsets $U \subset M$ is given by

$$\mu_U := \mathrm{e}^{-\lambda(U)} \sum_{n=0}^{\infty} \lambda_{U,n}. \tag{4}$$

The family of measures $(\mu_U)_{U \subset M}$ determines the Poisson probability measure μ on the manifold M [1].

3. Two-Rank Past and Preferred Past Structure

Classical and quantum fields on causal sets are described by discretized counterparts to the equations of motion on a continuum manifold. A recently proposed discretization method [3] needs a supplementary structure to a causal set called a *preferred past* as defined in the following.

A path from a causet event $x \in \mathscr{C}$ to an event in its future $y \succ x$ is the set of events $\mathscr{P}$ that forms the linked chain

$$x \prec\!\ast\, x_1 \prec\!\ast\, x_2 \prec\!\ast \cdots \prec\!\ast\, x_{n-2} \prec\!\ast\, y. \tag{5}$$

We denote the set of all *paths from* x *to* y by $\mathrm{paths}(x, y)$.

The length (number of links) of the shortest path is called the *rank*

$$\mathrm{rk}(y, x) := \begin{cases} \min\limits_{\mathscr{P} \in \mathrm{paths}(x,y)} |\mathscr{P}| - 1 & x \preceq y, \\ \infty & \text{otherwise}. \end{cases} \tag{6}$$

The k-*rank past* of an event $x \in \mathscr{C}$ is the set

$$R_k^-(x) := \{y \in \mathscr{C} \mid \mathrm{rk}(x, y) = k\}. \tag{7}$$

The *2-rank past infinity* C_2^- of a causet $\mathscr{C}$ is the set of events without any rank 2 past events. A *preferred past structure* is a choice of one event from the rank 2 past for every event $x \in \mathscr{C} \setminus C_2^-$ (that is not in the 2-rank past infinity C_2^-).

4. Diamonds

Let $x, y \in \mathscr{C}$ such that $\mathrm{rk}(y, x) = 2$. We call the Alexandrov set $I(x, y)$ a diamond with diamond size

$$k = |I(x, y) \setminus \{x, y\}|. \tag{8}$$

Diamonds of size k are also called k-diamonds. The events in the set $I(y, x) \setminus \{x, y\}$ can either be linked to x and y or they are related to other events in the diamond. The *number of internal events* is

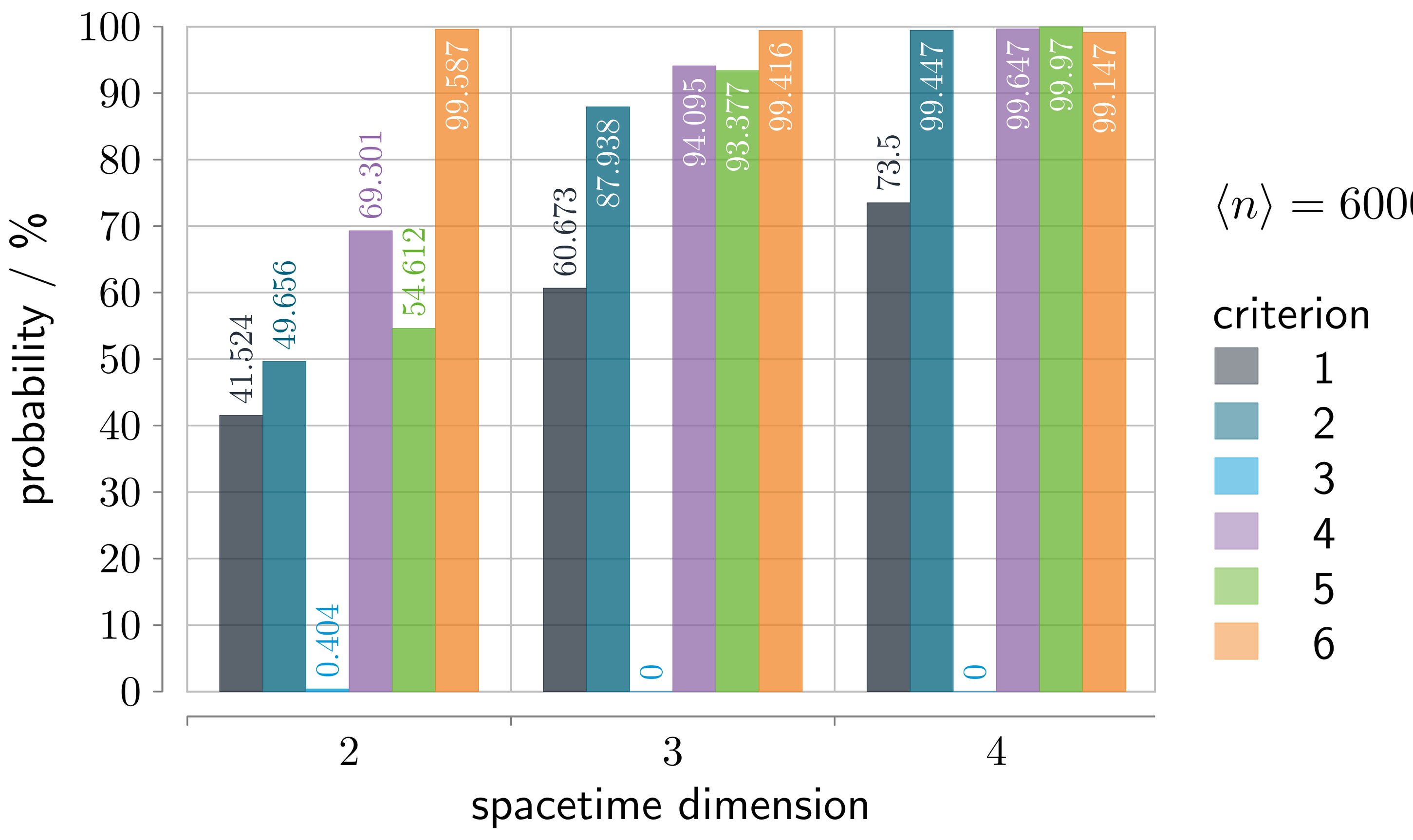
$$\mathrm{itn}(y, x) := k - |\{z \in I(x, y) \mid x \prec\!\ast\, z \prec\!\ast\, y\}|. \tag{9}$$

A diamond from y to x with $\mathrm{itn}(y, x) = 0$ is referred to as a *pure diamond*.

5. What Past to Prefer

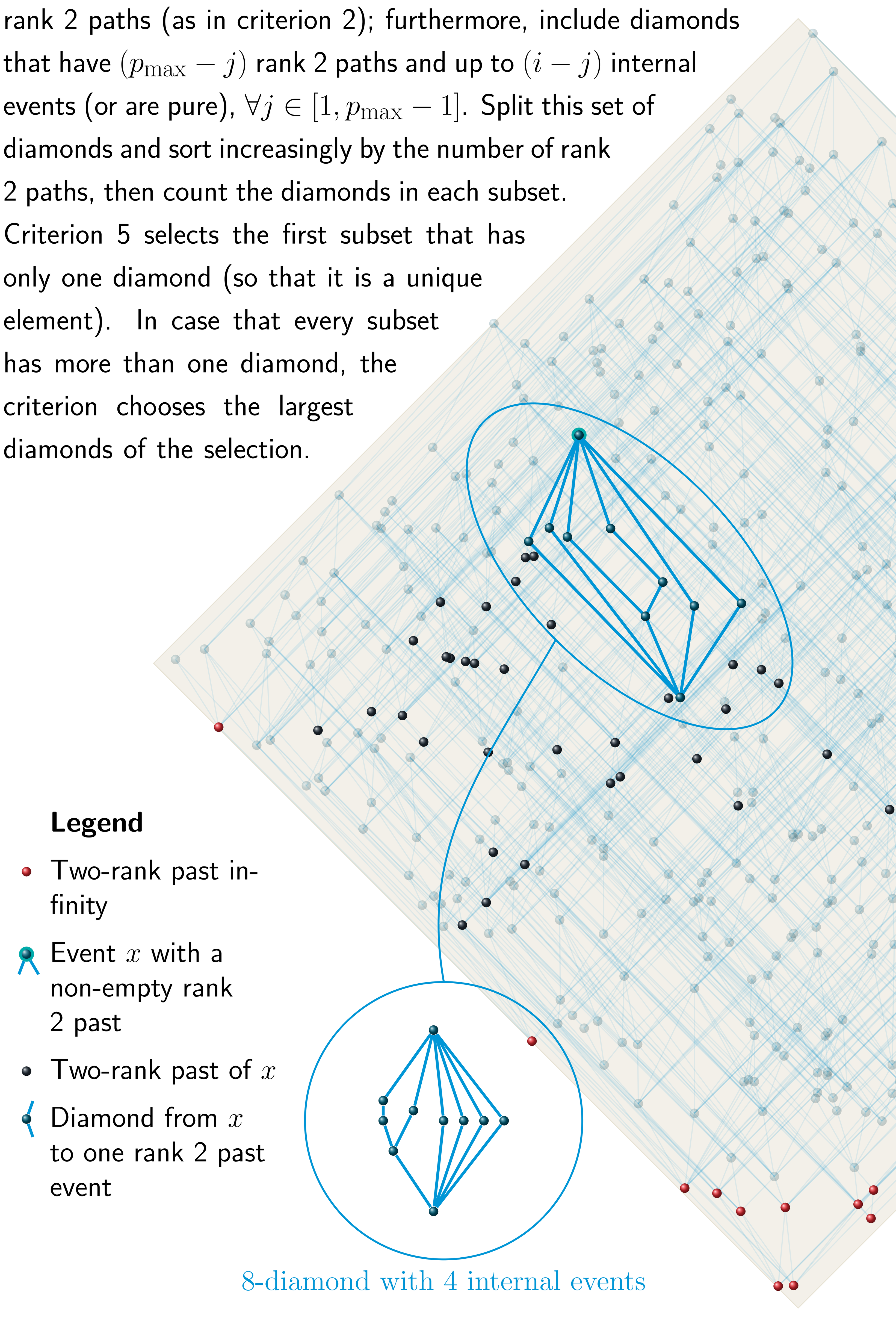
To determine a preferred past structure for a given causet, we numerically construct ensembles of 10,000 sprinkled causal sets in $1+1$, $1+2$ and $1+3$ dimensional Minkowski spacetimes with an expected number of 6,000 events, and we test six criteria in two qualities. We formulate the criteria in terms of the diamonds for an event $x \in \mathscr{C} \setminus C_2^-$ and one event in its rank 2 past $R_2^-(x)$. The criteria select the ◆ 1: largest pure diamonds, ◆ 2: diamonds with the least internal events of the diamonds with the most rank 2 paths, ◆ 3: smallest diamonds, ◆ 4: diamonds with the most internal events of the diamonds with the most rank 2 paths, ◆ 5: see description below, ◆ 6: largest diamonds.

Firstly, a good criterion should almost always select a unique event from the rank 2 past. The probability to obtain a unique event by our 6 criteria is shown in the graphic.



Secondly, we prefer small rank 2 past diamonds resulting in a small proper separation between the event and its preferred past. This condition is met by the criterion 1, 3 and 5. In combination of both qualities, criterion 5 turned out the best results:

◆ For criterion 5, consider all diamonds with the smallest number i of internal events of those diamonds with the greatest number $p_{\max}$ of rank 2 paths (as in criterion 2); furthermore, include diamonds that have $(p_{\max} - j)$ rank 2 paths and up to $(i - j)$ internal events (or are pure), $\forall j \in [1, p_{\max} - 1]$. Split this set of diamonds and sort increasingly by the number of rank 2 paths, then count the diamonds in each subset. Criterion 5 selects the first subset that has only one diamond (so that it is a unique element). In case that every subset has more than one diamond, the criterion chooses the largest diamonds of the selection.



8-diamond with 4 internal events

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