The Sprinkling Process, Diamonds and Local Structures in Causal Set Theory

Christoph Minz
(in joined work with Christopher J. Fewster, Eli Hawkins, and Kasia Rejzner)[1]

[1]Department of Mathematics
University of York

The Virtual Causet Conference, 27 August 2020

Notation

For all $x, y \in (\mathscr{S}, \preceq)$:

$$\text{Causal (Alexandrov) interval :} \quad I(x,y) := \{z \in \mathscr{S} | x \preceq z \preceq y\}$$
$$\text{Links :} \quad x \prec\!\ast\, y \Leftrightarrow (I(x,y) = \{x,y\} \text{ and } x \neq y)$$

What is Sprinkling?

Sprinkling is the Poisson process of obtaining a sprinkled causal set (*causet*) from a spacetime:

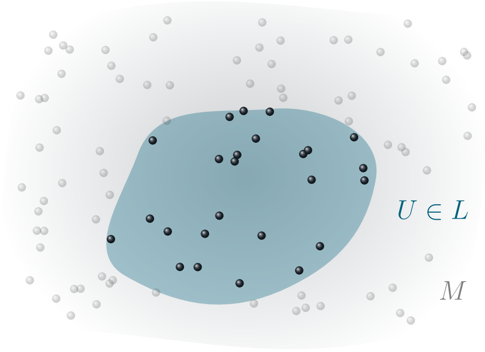
1. Let (M, g) be a smooth spacetime manifold with metric g.
2. Select a locally finite set called *sprinkle* of spacetime points $S \underset{\text{loc. finite}}{\subset} M$ by a Poisson process.
3. Restrict the causal relation from the spacetime to the sprinkle

$$x \preceq y \Leftrightarrow x \in J^{-}(y)$$

Let $U \in L$ be a subset of M in the set L of open, pre-compact subsets. The configuration spaces of *sprinkles* are

$$Q := \{S \subset M \mid \forall U \in L : |S \cap U| < \infty\},$$

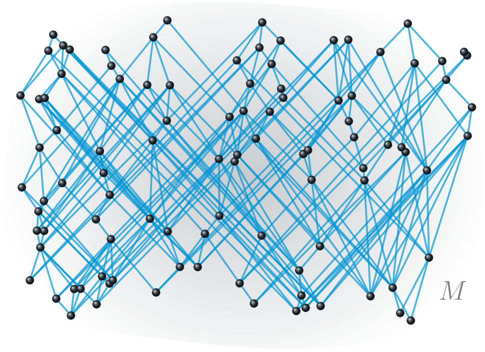
$$Q_{U,n} := \{S \subset U \mid |S| = n\}, \qquad Q_U := \bigcup_{n=0}^{\infty} Q_{U,n}$$



Poisson (probability) measure on U [AKR98] assigns a probability $\mu_U(B)$ to any element $B = (B_n)_{n \in \mathbb{N}_0}$ for measureable subsets $B_n \in \mathcal{B}(Q_{U,n})$.

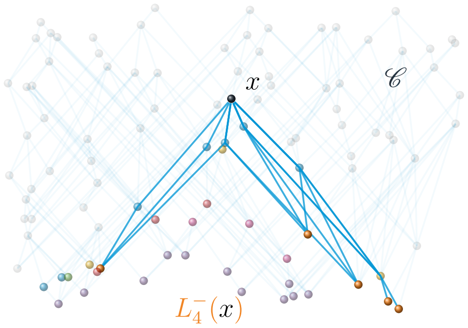
There exists a unique Poisson measure μ on the entire manifold M.

For Alexandrov subsets of $1+1$ dimensional Minkowski spacetime, the probability that a sprinkle is in a given causet isomorphism class can be computed from the number of distinct 2D orders.



- Classical and quantum fields on causal sets are described by discretized counterparts to the eom. on a continuum manifold.
- Most approaches to discretization of the Klein-Gordon equation are based on past *k-layers* [Sor09, ASS14],

$$L_k^-(x) := \{y \in \mathscr{C} \mid k = |I(y,x)| - 1\}.$$



- These discretizations need the spacetime dimension as an input, but in general the spacetime dimension of a given causal set is an emergent (local) property and not pre-defined.
- A more recently proposed discretization method [DHFRW20] has the potential to be independent of the dimension, but needs a supplementary structure to a causal set called a *preferred past* as defined in the following.
- We define a *path* from a causet event $x \in \mathscr{C}$ to an event in its future $y \succ x$ as the set of events $\mathscr{P}$ that forms the linked chain

$$x \prec\!\!* \, x_1 \prec\!\!* \, x_2 \prec\!\!* \cdots \prec\!\!* \, x_{n-2} \prec\!\!* \, y.$$

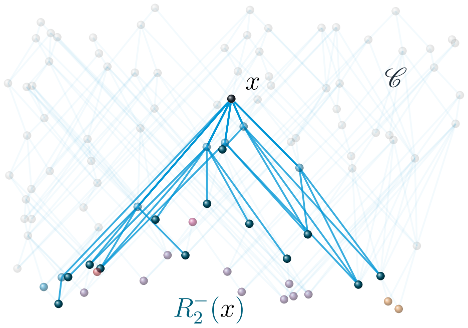
- We denote the set of all *paths from* x *to* y by $\mathrm{paths}(x, y)$.
- The length (number of links) of the shortest path is called the *rank*

$$\mathrm{rk}(y, x) := \begin{cases} \min\limits_{\mathscr{P} \in \mathrm{paths}(x,y)} |\mathscr{P}| - 1 & x \preceq y, \\ \infty & \text{otherwise.} \end{cases}$$

- The *rank k past* of an event $x \in \mathscr{C}$ is the set

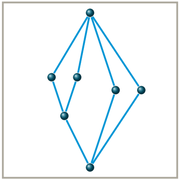
$$R_k^-(x) := \{y \in \mathscr{C} \mid \mathrm{rk}(x, y) = k\}.$$

- A *preferred past structure* is a map $\Lambda^- : \mathscr{C} \setminus C_2^- \to \mathscr{C}$ such that $\Lambda^-(x) \in R_2^-(x)$ for all $x \in \mathscr{C} \setminus C_2^-$.

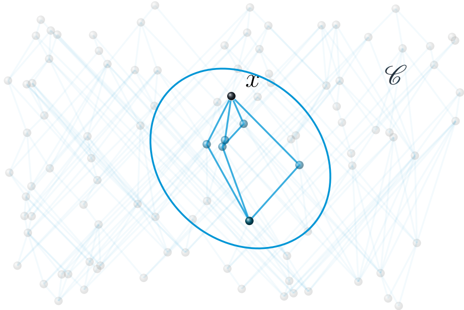


For an event $x \in \mathscr{C} \setminus C_2^-$ and $y \in R_2^-(x)$, we call the Alexandrov set $I(y, x)$ a *diamond* with the properties

$$
\begin{aligned}
\text{diamond } \textit{size}: &\quad k = |I(y,x) \setminus \{x, y\}|, \\
\#\ \textit{perimetral} \text{ events}: &\quad \mathrm{prm}(y,x) := \left|\{z \in I(x,y) \mid x \prec\!\ast\, z \prec\!\ast\, y\}\right|, \\
\#\ \textit{internal} \text{ events}: &\quad \mathrm{itn}(x,y) := k - \mathrm{prm}(y,x). \\
\text{It is } \textit{pure} \text{ if}: &\quad \mathrm{itn}(x,y) = 0.
\end{aligned}
$$



5-diamond
3 internal events



♦ 1: Rank 2 past events that span the largest diamonds (“maximal layer rule” proposed in [DHFRW20]),

$$D^{-}_{\mathrm{crit}\,1}(x) := \underset{y \in R^{-}_{2}(x)}{\arg\max}\, |I(y,x)|.$$

mostly singleton
arbitrarily large proper time separation

♦ 2: Rank 2 past events that span the smallest diamonds,

$$D^{-}_{\mathrm{crit}\,2}(x) := \underset{y \in R^{-}_{2}(x)}{\arg\min}\, |I(y,x)|.$$

smallest proper time separation
almost never singleton

♦ 3: Among all rank 2 past events that span pure diamonds,

$$D^{-}_{\mathrm{pure}}(x) := \left\{ y \in R^{-}_{2}(x) \mid \mathrm{itn}(x,y) = 0 \right\},$$

consider the largest (or maximal perimetral) pure diamonds,

$$D^{-}_{\mathrm{crit}\,3}(x) := \underset{y \in D^{-}_{\mathrm{pure}}(x)}{\arg\max}\, |I(y,x)| = \underset{y \in D^{-}_{\mathrm{pure}}(x)}{\arg\max}\, \mathrm{prm}(x,y)$$

small proper time separation; not always singleton

Consider the rank 2 past events that span diamonds with the maximal number of perimetral events

$$D^-_{\max \mathrm{prm}}(x) := \underset{y \in R_2^-(x)}{\arg\max}\, \mathrm{prm}(x, y).$$

♦ 4: Diamonds with the most internal events among the diamonds with the most perimetral events,

$$D^-_{\mathrm{crit}\,4}(x) := \underset{y \in D^-_{\max \mathrm{prm}}(x)}{\arg\max}\, \mathrm{itn}(x, y).$$

often singleton
big proper time separation

♦ 5: Diamonds with the least internal events among the diamonds with the most perimetral events,

$$D^-_{\mathrm{crit}\,5}(x) := \underset{y \in D^-_{\max \mathrm{prm}}(x)}{\arg\min}\, \mathrm{itn}(x, y).$$

small proper time separation
mostly singleton only in dimension $1 + 3$

6: Select the same subset as criterion 5 when there are no singletons among the sets classified by the number of internal events $i \in \mathbb{N}_0$ and the number of perimetral events $p \in \mathbb{N}$,

$$D^-_{i,p}(x) := \left\{ y \in R^-_2(x) \;\middle|\; \mathrm{itn}(x,y) = i, \mathrm{prm}(x,y) = p \right\}.$$

If there is at least one singleton in this matrix, then choose

$$j(x) := \min \left\{ i \;\middle|\; |D^-_{i,p}(x)| = 1 \right\}, \quad q(x) := \max \left\{ p \;\middle|\; |D^-_{j(x),p}| = 1 \right\},$$

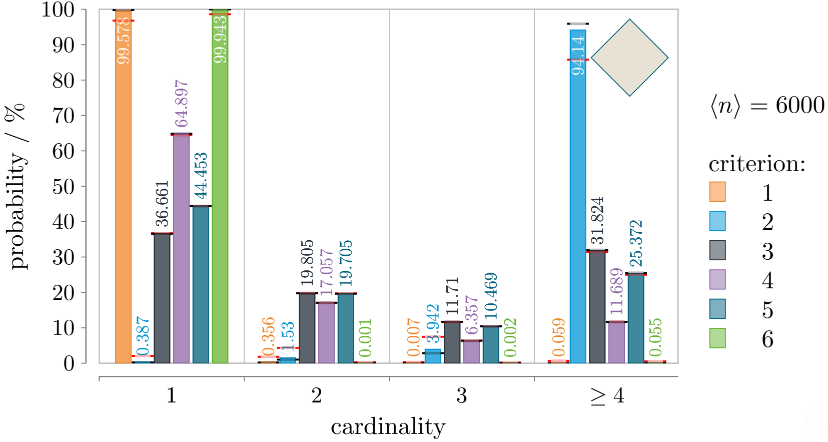
so that

$$D^-_{\mathrm{crit}\,6}(x) := \begin{cases} D^-_{\mathrm{crit}\,5}(x), & \text{if there are no singletons among } D^-_{i,p}(x), \\ D^-_{j(x),q(x)}(x), & \text{if } j(x) < \infty. \end{cases}$$

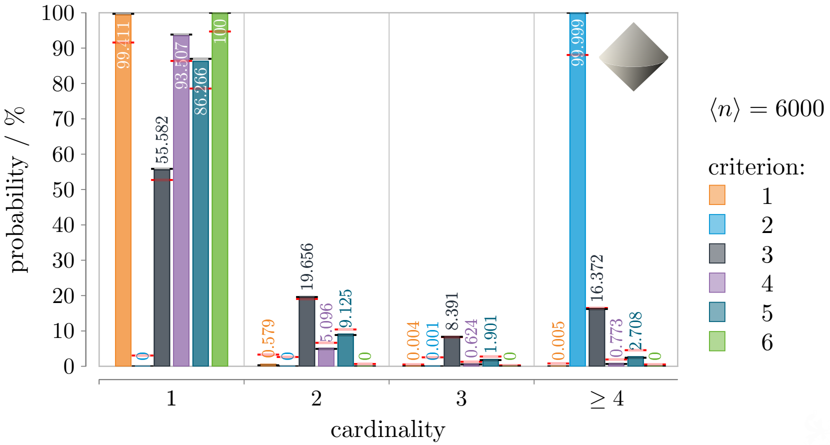
Diamond Count Matrix

i \ p	1	2	3	4	5	
	1 ♦ 1	0	0	0	0	0
9	1	0	0	0	0	0
8	2	0	0	0	0	0
7	3	1	0	0	0	0
6	1	0	1	0	0	0
5	2	2	1	0	0	0
4	7	2	0	1	1 ♦ 4	0
3	6	3	1	0	2 ♦ 5	0
2	4	1	2	1 ♦ 6	0	0
0	6 ♦ 2	0	2 ♦ 3	0	0	0

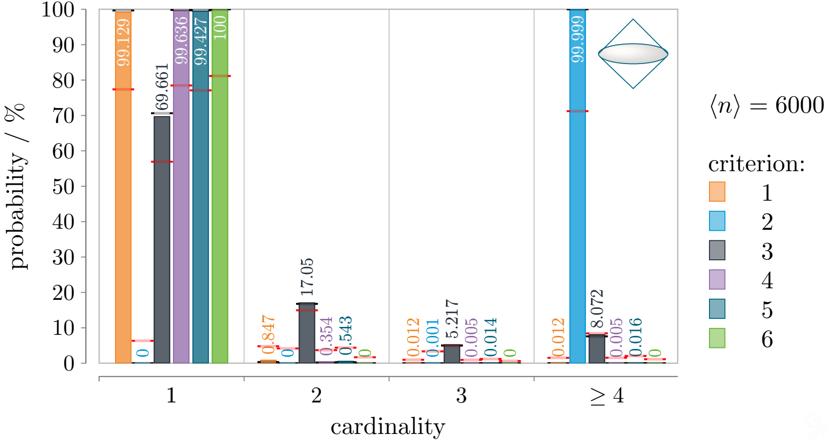
Number of events in the rank 2 past selected by each criterion, considering 10,000 sprinkles in an Alexandrov subset of flat spacetime in dimension $\underbrace{1+1}$, $1+2$, $1+3$



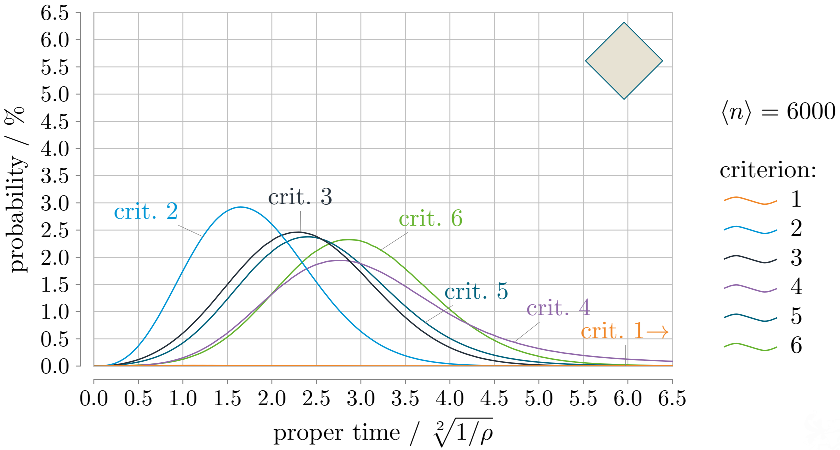
Number of events in the rank 2 past selected by each criterion, considering 10,000 sprinkles in an Alexandrov subset of flat spacetime in dimension $1+1$, $\underbrace{1+2}$, $1+3$



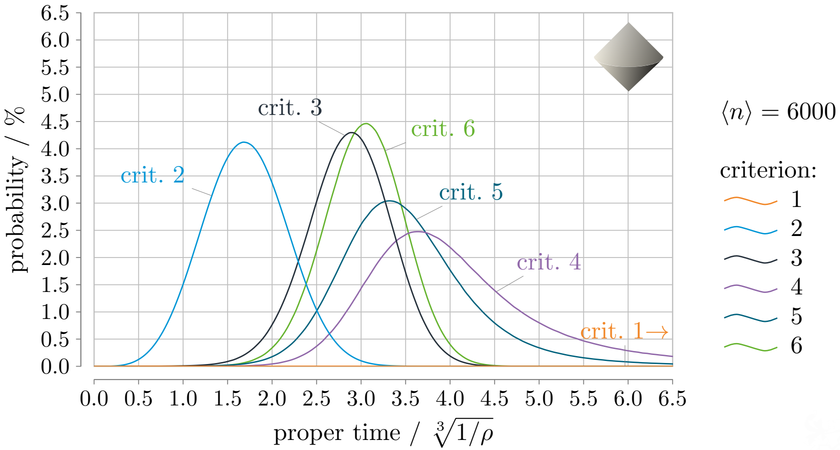
Number of events in the rank 2 past selected by each criterion, considering 10,000 sprinkles in an Alexandrov subset of flat spacetime in dimension $1+1$, $1+2$, $\underbrace{1+3}$



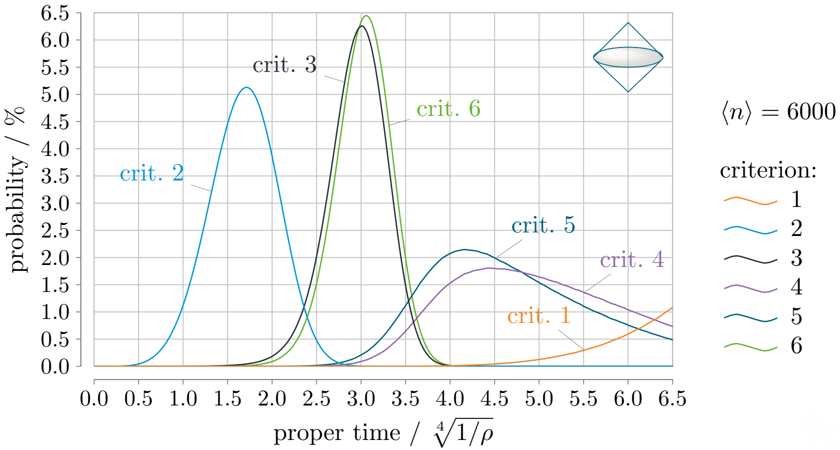
Proper time separation between the past and future event of the diamonds, considering 10,000 sprinkles in an Alexandrov subset of flat spacetime in dimension $1+1$, $\underbrace{1+2}$, $1+3$



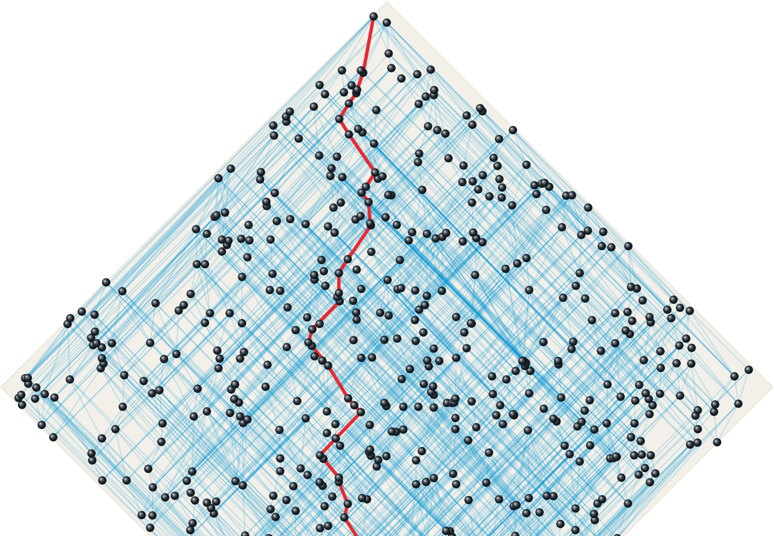
Proper time separation between the past and future event of the diamonds, considering 10,000 sprinkles in an Alexandrov subset of flat spacetime in dimension $\underbrace{1+1}$, $1+2$, $1+3$



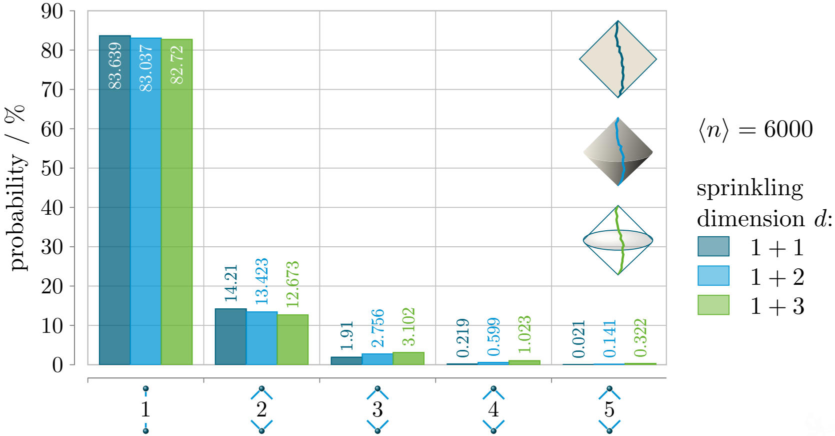
Proper time separation between the past and future event of the diamonds, considering 10,000 sprinkles in an Alexandrov subset of flat spacetime in dimension $1+1$, $1+2$, $\underbrace{1+3}$



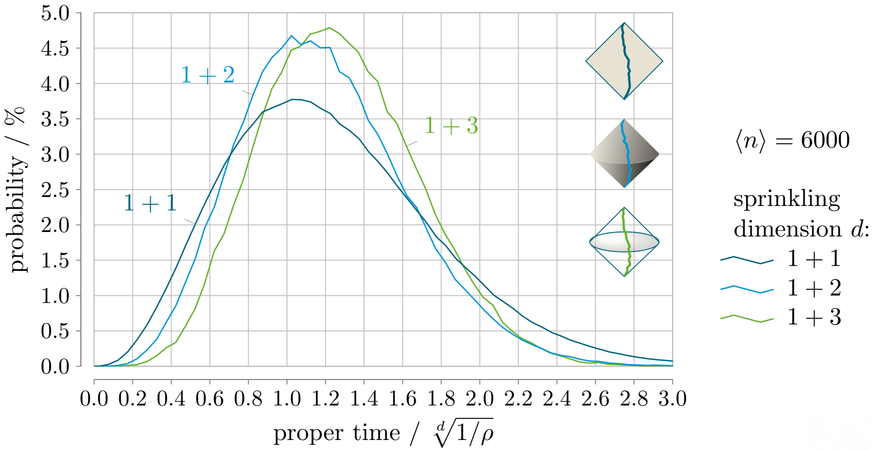
Now consider diamonds between next to nearest neighbours along the timelike geodesic path (longest path) between the events with the smallest and largest time coordinate,



Diamond size distribution, considering 10,000 sprinkles in an Alexandrov subset of flat spacetimes, for expected geodesic path lengths: 137.4 in $1+1$, 26.2 events in $1+2$, and 8.1 events in $1+3$.



Proper time separation distributions, considering 10,000 sprinkles in an Alexandrov subset of flat spacetimes,



Sergio Albeverio, Yu G Kondratiev, and Michael Röckner.
Analysis and Geometry on Configuration Spaces.
Journal of Functional Analysis, 154:444–500, 1998.

Siavash Aslanbeigi, Mehdi Saravani, and Rafael D. Sorkin.
Generalized Causal Set d'Alembertians.
Journal of High Energy Physics, 2014(6):24, Jun 2014.

Edmund Dable-Heath, Christopher J Fewster, Kasia Rejzner, and Nick Woods.
Algebraic Classical and Quantum Field Theory on Causal Sets.
Physical Review D, 101(6):065013, 2020.

Rafael D. Sorkin.
Does Locality Fail at Intermediate Length-Scales.
Approaches to Quantum Gravity, pages 26–43, 2009.