Geometric construction of distinguished states

by Christoph Minz (in joint work with Eli Hawkins and Kasia Rejzner)*

UNIVERSITY of York

General ideas

- symplectic manifold with Riemannian metric $(\mathcal{M}, \omega, g)$ (phase space of classical fields and momenta)
- classical C*-algebra $\mathfrak{A}_0$ and a dense Poisson subalgebra $\mathcal{A}_0 \subseteq \mathfrak{A}_0$ of functionals over the manifold

Geometric quantization

For each $\hbar \in I_* \subset \mathbb{R}_+ \setminus \{0\}$, a quantization bundle is a Hermitian line bundle $\mathcal{L}_\hbar \to \mathcal{M}$ with connection $\nabla_\hbar$ such that

$$\operatorname{curv}(\nabla_\hbar) = -\frac{\mathrm{i}}{\hbar}\omega.$$

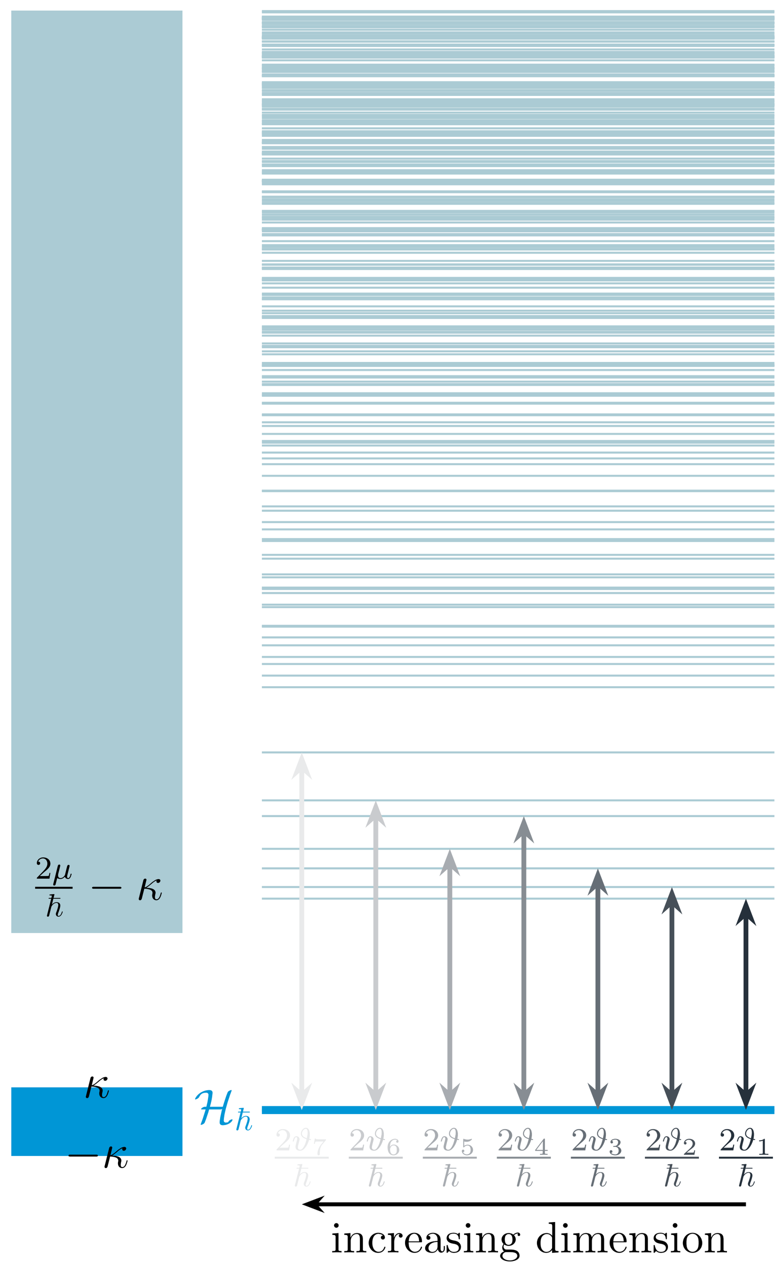
The Bochner Laplacian $\triangle_\hbar$ is an unbounded operator on sections $\Gamma(\mathcal{M}, \mathcal{L}_\hbar)$ determined by the connection $\nabla_\hbar$ and metric g,

$$\triangle_\hbar = \nabla_\hbar^* \nabla_\hbar.$$

In [1], it was shown that for $\kappa, \mu > 0$ (independent of $\hbar$) the (renormalized) Bochner Laplacian fulfills

$$\operatorname{spec}(\triangle_{\hbar,\Phi}) \subset [-\kappa, \kappa] \cup \left[\frac{2\mu}{\hbar} - \kappa, \infty\right).$$

As physical Hilbert space $\mathcal{H}_\hbar \subset \mathrm{L}^2(\mathcal{M}, \mathcal{L}_\hbar)$, consider the span of the sections corresponding to the lower part of the spectrum.



The Berezin-Toeplitz quantization and dequantization

The projector $\Pi_\hbar : \mathrm{L}^2(\mathcal{M}, \mathcal{L}_\hbar) \to \mathcal{H}_\hbar$ determines a quantization map

$$T_\hbar : \mathcal{A}_0 \to \mathcal{B}(\mathcal{H}_\hbar),$$

such that $\forall \psi \in \mathcal{H}_\hbar : T_\hbar(f)\psi = \Pi_\hbar(f\psi)$.

Assuming that Toeplitz operators of compactly supported functions $f \in \mathrm{C_c}(\mathcal{M}, \mathbb{C})$ are of trace-class such that a measure $\mu_\hbar$ exists,

$$\operatorname{Tr}(T_\hbar(f)) = \int_\mathcal{M} f \,\mathrm{d}\mu_\hbar.$$

The (Berezin)-Toeplitz dequantization is a family of linear maps

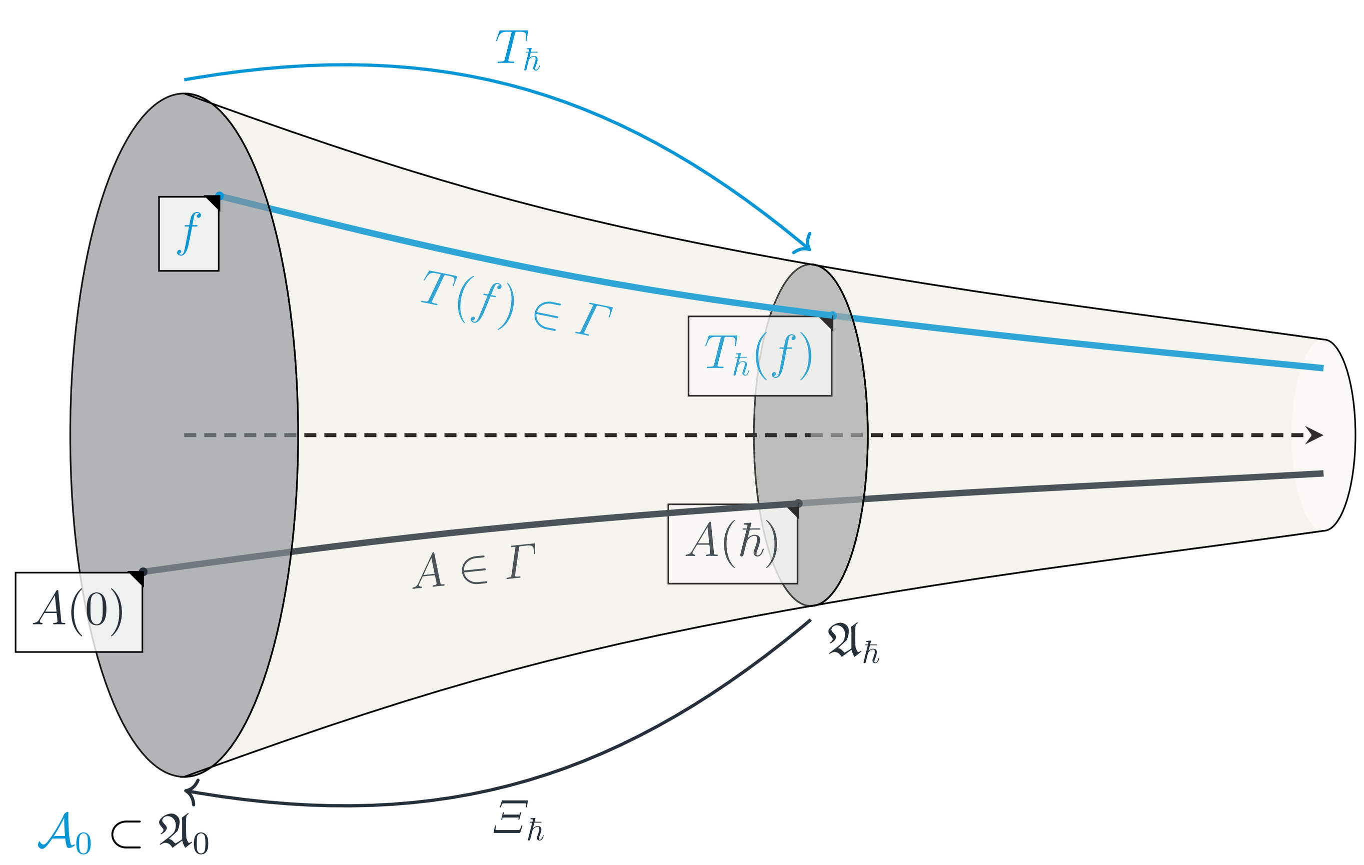
$$\Xi_\hbar : \mathfrak{A}_\hbar \to \mathfrak{A}_0,$$

such that for all operators $A \in \mathfrak{A}_\hbar$

$$\operatorname{Tr}(A T_\hbar(f)) = \int_\mathcal{M} \Xi_\hbar(A) f \,\mathrm{d}\mu_\hbar.$$

Relationship to strict deformation quantization

Under certain conditions, the quantized observables can be turned into sections Γ of a continuous field of C*-algebras $(I, (\mathfrak{A}_\hbar)_{\hbar \in I}, \Gamma)$ over a range I of quantization parameters, including the classical limit at $\hbar = 0$ [2].



Explicit construction in finite dimensions

- free scalar fields and momenta on a (subset of a) causal set [3, 4]
- determine a $2N$-dimensional symplectic vector space (solution space) with inner product $(\mathcal{S}, \omega, \langle \cdot, \cdot \rangle)$

Spectrum of the Bochner Laplacian

The symplectic form ω on $\mathcal{S}$ may be expressed by the inverse of the (restricted) Pauli-Jordan operator $E \in \operatorname{End}(\mathcal{S})$ and the inner product,

$$\forall v_1, v_2 \in \mathcal{S} : \quad \omega(v_1, v_2) = \left\langle v_1, E^{-1} v_2 \right\rangle.$$

This relation determines some positive numbers ϑ_i such that the spectrum of the Bochner Laplacian $\triangle_\hbar$ is

$$\operatorname{spec}(\triangle_\hbar) = \left\{ \frac{1}{\hbar} \sum_{i=1}^{N} (2n_i + 1)\vartheta_i \,\middle|\, n_i \in \mathbb{N} \right\}.$$

Here the Hilbert space $\mathcal{H}_\hbar$ is spanned by all sections corresponding to the smallest eigenvalue. The Berezin-Toeplitz quantization and dequantization maps follow the general idea (left column).

The dequantization state

A linear functional $\sigma : \mathfrak{A}_\hbar \to \mathbb{C}$ is a state if and only if it is positive,

$$\forall A \in \mathfrak{A}_\hbar : \qquad \sigma(A^* A) \geq 0,$$

and has unit norm. We showed that the linear map $\sigma_\hbar : \mathfrak{A}_\hbar \to \mathbb{C}$ given by

$$\sigma_\hbar(A) := \Xi_\hbar(A)(0)$$

is a state.

For a Weyl operator $W_\hbar(\phi)$ of any covector $\phi \in \mathcal{S}^* = \operatorname{Hom}(\mathcal{S}, \mathbb{R})$ the dequantization is

$$\sigma_\hbar(W_\hbar(\phi)) = \exp\left(-\frac{\hbar}{2}|\phi|^2\right).$$

For any Toeplitz operator $T_\hbar(f) \in \mathfrak{A}_\hbar$, the Sorkin-Johnston state $\sigma_\hbar$ is the Berezin transform

$$b_\hbar(z) := \frac{1}{(2\pi\hbar)^N} \exp\left(-\frac{1}{\hbar}|z|^2\right).$$

of $f \in \mathcal{A}_0$ evaluated at 0,

$$\sigma_\hbar(T_\hbar(f)) = \int_\mathcal{S} b_\hbar(z) f(z) \,\mathrm{dvol}(z).$$

Equivalence with the Sorkin-Johnston state

Let $\mathcal{W}_\hbar(\mathcal{S}^*)$ be the Weyl algebra spanned by the Weyl generators $W_\hbar(\phi)$. The Sorkin-Johnston state $\sigma_{\mathrm{SJ}} : \mathcal{W}_\hbar(\mathcal{S}^*) \to \mathbb{C}$ is the quasi-free (or Gaussian) state with a covariance given by the inverse of the symmetric, bi-linear form η,

$$\forall v_1, v_2 \in \mathcal{S} : \qquad \eta(v_1, v_2) := \left\langle v_1, |E|^{-1} v_2 \right\rangle.$$

The dequantization state is the same state.

Please find more details here.
preprint 2207.05667

https://arxiv.org/abs/2207.05667

[1] X Ma, G Marinescu. *Mathematische Zeitschrift*, 240(3):651-664, 2002.
[2] J Dixmier. Les C*-algèbres et Leurs Représentations. 1964.
[3] J Henson. The Causal Set Approach to Quantum Gravity. 393, 2009.
[4] E D-H, C J Fewster, K Rejzner, and N Woods. *Physical Review D*, 101(6):065013, 2020.

These are results of my PhD research at the University of York funded by ESPRC (grand number EP/N509802/1).

* E-mail: christoph.minz@itp.uni-leipzig.de