From classical to quantum fields on causal sets

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Causal sets for quantum gravity (QG)

As a framework for QG, causal set theory (CST) takes a minimalistic perspective, reducing the spacetime manifold structure to:

- a causal relation between elements, which is a partially ordered set (poset), and
- a volume scale, which is identified through some fundamental lenth scale.

This is enough to recover the full structure of a spacetime, see review [Surya 2019]. Additionally

- causal sets are locally finite, and
- spacetime manifolds are conjectured to be continuous approximations to causal sets [Bombelli-Lee-Meyer-Sorkin 1987].

Causal sets as discrete models

Discreteness $\Rightarrow$ finite-dimensional analogoues of classical and quantum field theory.

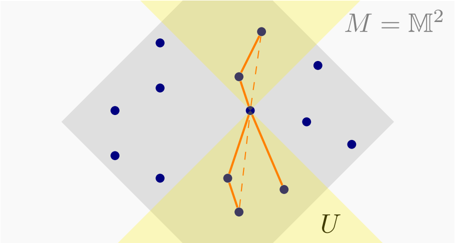
- simpler to treat than the infinite-dimensional spaces
- but require a limit procedure to find corresponding expressions that are valid for the infinite-dimensional settings.

Topics of this talk

1. Causal sets, classical fields and their discretisation
2. Quantum algebras and their relation to the classical theory
3. Geometric quantization and dequantization
4. Summary and generalizations to interacting field theory

Based on my PhD research with Eli Hawkins and Kasia Rejzner at the Department of Mathematics, University of York.

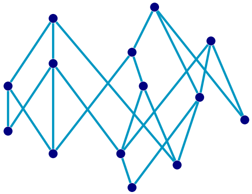
A sprinkle on a spacetime M



- Probability space $\left(Q_U, \mathcal{B}(Q_U), \mu_U\right)$
- $Q_{U,n} := \{S \subset U \mid |S| = n\}$
- $\mu_U(B_n) = \mathrm{e}^{-\rho\nu(U)} \frac{\rho^n}{n!} \nu^n \left(\Sigma_{U,n}^{-1}(B_n)\right)$

Math. review: [Fewster-Hawkins-Minz-Rejzner 2021].

Sprinkled causal set



drawn with the LaTeX-package **causets**:
$\backslash\text{pcauset}\{12, 7, 4, 11, 5, 14, 9, 2, 10, 13, 1, 6, 3, 8\}$

The Hasse diagram of a causal set C has

- a vertex for every element
- an edge between every pair of elements $a < b$ if $\nexists c \in C : a < c < b$

Definition (Causal set)

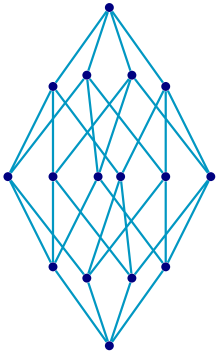
A *causal set* (causet) is a partially ordered set $(S, \preceq)$ that is locally finite, i.e. the cardinality of the interval between any two elements (events) $x, y \in S$,

$$[x, y] := \{z \in S \mid x \preceq z \preceq y\},$$

is finite.

Similarities with spacetime manifolds

- causal structure is a partial order,
- local compactness of a spacetime manifold $\Rightarrow$ local finiteness of a causet model of this manifold



A finite causet (3-simplex) that embeds in d-dimensional Minkowski spacetime with $d \geq 1 + 3$.

Definition (Scalar field configuration space)

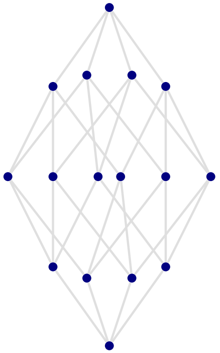
For (a local region of) a causet C, the configuration space is given by all functions $\mathcal{E}(C) := \{\varphi : C \to \mathbb{R}\}$.
For a spacetime manifold M, the space of real scalar fields is the space of smooth functions, $\mathcal{E}(M) := \mathrm{C}^\infty(M, \mathbb{R})$.

Definition (Algebra of classical observables)

The algebra of classical observables on $\mathcal{X} = C$ or $\mathcal{X} = M$ is the space of smooth, complex-valued functionals over the configuration space $\mathcal{F}(\mathcal{X}) = \mathrm{C}^\infty\big(\mathcal{E}(\mathcal{X}), \mathbb{C}\big)$ with pointwise addition and multiplication.

(Off-shell) Poisson bracket [DableHeath-Fewster-Rejzner-Woods 2020]:

$$\{f_1, f_2\}(\varphi) = \pi_{\mathrm{off}}\Big(f_1'(\varphi), f_2'(\varphi)\Big).$$



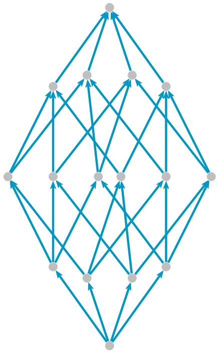
For a finite causet C with cardinality n, $\mathcal{E}(C) \cong \mathbb{R}^n$, write $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)^{\mathsf{T}}$.

Imposing field equations

To discretize the Klein-Gordon equations $P\varphi = 0$, consider

$$(P\varphi)(x) = c_0\varphi(x) + c_1 \sum_{z \in P_1(x)} \varphi(z) + c_2 \sum_{z \in P_2(x)} \varphi(z) + \dots$$

where $P_i(x)$ are subsets of events in the past of x.
Disadvantage of the layer method: the coefficients c_i depend on the spacetime, especially its dimension.
Alternative approach: a preferred past structure that assigns a unique (rank 2) past element y to every element x, and $(P\varphi)(x)$ depends only on the field values of $[y, x]$ [DableHeath-Fewster-Rejzner-Woods 2020]. There are ways to define a pref. past structure for a given causal set intrinsically, [Fewster-Hawkins-Minz-Rejzner 2021, Minz 2022].
Open question about this alternative: Does it approximate the continuum operator well?



In common discretization methods, $P_i(x)$ are past layers $L_i^-(x)$ [Dowker-Glaser 2013].

Green operators

The matrix of the field operator P is invertible.
So we define

retarded Green operator: $E^{+} := P^{-1}$,

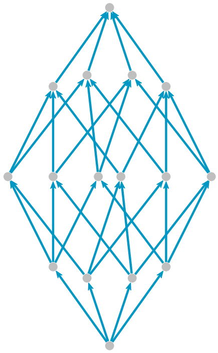
advanced Green operator: $E^{-} := (E^{+})^{*} = (E^{+})^{\mathsf{T}}$,

Pauli-Jordan operator: $E := E^{+} - E^{-}$,

with respect to the standard inner product (metric).

The solution space

For spacetime manifolds: $\mathcal{S} = \ker(P) = \operatorname{img}(E)$.
For causal sets: $\mathcal{S} = \operatorname{img}(E) \supset \ker(P) = \{0\}$.
With E, we get the Peierls bracket $\{f, g\} = \dfrac{\partial f}{\partial x_i} E^{ij} \dfrac{\partial g}{\partial x_j}$ [Peierls 1997]



Quantum-classical correspondence

A quantum theory has to be compatible with the corresponding classical theory, meaning that it should reproduce it in the limit $\hbar \to 0$.
Let $(\mathfrak{A}_\hbar, \bullet_\hbar)$ be the algebra of the quantum theory (at $\hbar$). A classical theory $(\mathfrak{A}_0, \cdot)$ corresponds to it if the product turns into the commutative one and the commutator turns into the Poisson bracket,

$$\lim_{\hbar \to 0} f_\hbar \bullet_\hbar g_\hbar = f \cdot g,$$

$$\lim_{\hbar \to 0} \frac{1}{\mathrm{i}\hbar} \Big(f_\hbar \bullet_\hbar g_\hbar - g_\hbar \bullet_\hbar f_\hbar \Big) = \{f, g\},$$

where the observables may depend on $\hbar$ as well, $\lim_{\hbar \to 0} f_\hbar = f$, $\lim_{\hbar \to 0} g_\hbar = g$.

Definition (Quantization)

A quantization is a family of maps $Q_\hbar : \mathcal{A}_0 \to \mathfrak{A}_\hbar$ (for some *-subalgebra $\mathcal{A}_0 \subseteq \mathfrak{A}_0$) such that $Q_\hbar(f)^* = Q_\hbar(\overline{f})$ (and the unit is preserved).

Can we find quantization maps?

Yes, but:
By the Groenewald-van Hove no-go theorem [1946, 1951], there is no quantization map such that

$$[Q_\hbar(f), Q_\hbar(g)]_- = \mathrm{i}\hbar Q_\hbar\big(\{f, g\}\big).$$

In general, there must be higher order correction terms in $\hbar$ on the right.

Formal deformation quantization

Extend the classical Poisson algebra $(\mathcal{P}, \{\cdot, \cdot\})$ to formal power series $\mathcal{P}[[\hbar]] \ni f, g, h$ with a star product

$$f \star g = \sum_{k=0}^{\infty} B_k(f, g)\hbar^k, \text{ with}$$
$$(f \star g) \star h = f \star (g \star h),$$
$$B_0(f, g) = f \cdot g,$$
$$B_1(f, g) - B_1(g, f) = \mathrm{i}\{f, g\},$$

(and perhaps some other properties).
Note that not all bilinear maps B_k are fixed, so there is no unique star product.

Wick products

For some matrix W, and pointwise multiplication $m : f \otimes g \mapsto f \cdot g$:

$$f \star_H g = m \circ \mathrm{e}^{\hbar W^{ij} \partial_i \otimes \partial_j}(f \otimes g),$$
$$W = \frac{\mathrm{i}}{2} E + H,$$

where

- $\mathrm{Re}(W) = H$
- W is positive semi-definite, $W \geq 0$
- $\ker W \subseteq \ker E$

W has the meaning of a 2-point function.
On a spacetime, the Wick products fulfill the equal-time commutation relations.

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Sorkin-Johnston 2-point function

[Johnston 2010, Sorkin 2011, 2017] used the axioms

$$\begin{aligned} \text{commutator:} \quad & W_{\mathrm{SJ}} - \overline{W}_{\mathrm{SJ}} = \mathrm{i}E, \\ \text{positivity:} \quad & W_{\mathrm{SJ}} \geq 0, \\ \text{purity:} \quad & \overline{W}_{\mathrm{SJ}} W_{\mathrm{SJ}} = 0. \end{aligned}$$

$$\Rightarrow W_{\mathrm{SJ}} = \frac{\mathrm{i}}{2} E + \sqrt{-E^2}$$

- For spacetimes, the state is not Hadamard [Fewster-Verch 2012, 2013], but (non-unique) modifications of the inner product make it Hadamard [Brum-Fredenhagen 2014, Wingham 2019].
- A similar construction can be done for fermions [Finster 2005–2011, Fewster-Lang 2015].

It is convinient to define quasifree states using the Weyl generators.

Definition (Weyl quantization)

The Weyl algebra $\mathfrak{W}_\hbar$ over a real vector space $\mathcal{S}$ with a Poisson bracket $\{\cdot, \cdot\}$ for the linear observables in $\mathcal{S}^* = \mathrm{Hom}(\mathcal{S}, \mathbb{R})$ is generated by the image of the map $W_\hbar : \mathcal{S}^* \to \mathfrak{W}_\hbar$ such that

$$W_\hbar(f) W_\hbar(g) = \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}\{f,g\}} W_\hbar(f+g),$$
$$W_\hbar(f)^* = W_\hbar(-f),$$
$$W_\hbar(0) = \mathbb{1}.$$

The map $W_\hbar$ is a quantization.

Definition (Algebraic state)

A state is a linear functional $\sigma : \mathfrak{A}_\hbar \to \mathbb{C}$ that is positive ($\forall A \in \mathfrak{A}_\hbar : \sigma(A^* A) \geq 0$) and normalized.

Definition (Quasi-free state)

A state σ is called quasi-free (or Gaussian) if there exists a symmetric, bi-linear form γ (the covariance of the state) on $\mathcal{S}^*$ such that

$$\sigma\big(W_\hbar(f)\big) = \exp\left(-\frac{\hbar}{4}\gamma(f,f)\right)$$

holds for the Weyl generator $W_\hbar(f)$ of every element $f \in \mathcal{S}^*$.

For the SJ construction: $\gamma^{-1}(\varphi, \varphi) = \langle \varphi, |E|^{-1} \varphi \rangle$

Quantization methods

- Formal deformation quantization – via star products (for causal sets, see [DableHeath-Fewster-Rejzner-Woods 2020])
- Strict deformation quantization – via a field of C*-algebras
- Weyl quantization
- Geometric quantization
  1. via a quantization line bundle
  2. the Bochner Laplacian and
  3. the Toeplitz quantization map

Even though, quantization is usually not invertible, some quantizations admit a dual map.

Definition (Dequantization)

A dequantization Υ is a family of linear maps

$$\Upsilon_\hbar : \mathfrak{A}_\hbar \to \mathfrak{A}_0,$$

that respects involution, $\Upsilon_\hbar(A^*) = \overline{\Upsilon_\hbar(A)}$, and if there exists a unit $\mathbf{1} \in \mathfrak{A}_\hbar$, it is also unital, $\Upsilon_\hbar(\mathbf{1}) = 1$.

So, for a quantization Q and a dequantization Υ, $\Upsilon_\hbar \circ Q_\hbar$ is usually not the identity map.

Idea: construct a state with a dequantization map.

Structure for geometric quantization

- a $2N$-dimensional vector space $\mathcal{S}$ with
- an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}$, and
- a symplectic form ω as inverse of the non-degenerate, on-shell Poisson bracket

such that

$$\forall \varphi_1, \varphi_2 \in \mathcal{S} : \quad \omega(\varphi_1, \varphi_2) = \left\langle \varphi_1, E^{-1} \varphi_2 \right\rangle ,$$

(E is closely related to the Pauli-Jordan operator)

Definition (Quantization bundle)

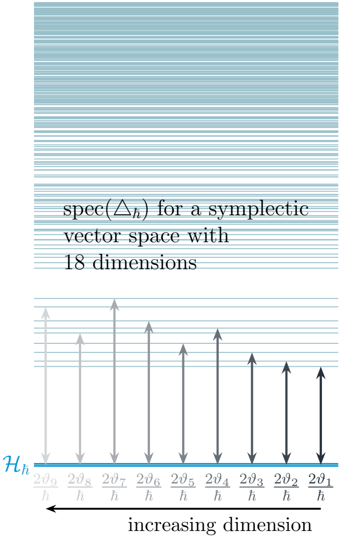
Let $(\mathcal{M}, \omega)$ be a real, symplectic manifold. A quantization bundle is a Hermitian line bundle $\mathcal{L}_\hbar \to \mathcal{M}$ with connection $\nabla_\hbar$ such that

$$\mathrm{curv}(\nabla_\hbar) = -\frac{\mathrm{i}}{\hbar}\omega.$$

Physical Hilbert space = subbundle

A physical Hilbert space $\mathcal{H}_\hbar$ is constructed from a subbundle (polarized sections).
Consider the eigensections of the lowest spectral part of the Bochner Laplacian

$$\triangle_\hbar := \nabla_\hbar^* \nabla_\hbar .$$



Theorem (Spectrum of the Bochner Laplacian)

The spectrum of the Laplacian for the quantization bundle $\mathcal{L}_\hbar \to \mathcal{S}$ over $(\mathcal{S}, \omega, \langle \cdot, \cdot \rangle)$ is determined by a set of strictly-positive numbers $\vartheta_i \in \mathbb{R}$ such that

$$\mathrm{spec}(\triangle_\hbar) = \left\{ \frac{1}{\hbar} \sum_{i=1}^{N} (2n_i + 1)\vartheta_i \;\middle|\; n_i \in \mathbb{N} \right\}.$$

In the figure: spectrum for a $2N$-dimensional space (with $N \in [1, 9]$) and bounds in more general cases of some symplectic manifolds [Ma-Marinescu 2002, 2008].

Physical Hilbert space $\mathcal{H}_\hbar$ from sections of the lowest part of the spectrum (here, a single eigenvalue)

Coordinates for the finite-dim. vector space $(z^i, \overline{z}^{\overline{\imath}})$. The holomorphic sections

$$\psi(z) = \frac{\alpha(z)}{\sqrt{2\pi\hbar}^N} \exp\left(-\frac{1}{2\hbar}|z|^2\right)$$

with any holomorphic function α describe the Hilbert space, using an orthonormal basis $|n_1, \dots, n_N\rangle$.

$$a^+_{\overline{\imath}} := \frac{1}{\sqrt{\hbar}}\delta_{\overline{\imath}i} z^i - \sqrt{\hbar}\nabla_{\overline{\imath}},$$
$$a^-_i := \frac{1}{\sqrt{\hbar}}\delta_{i\overline{\imath}} \overline{z}^{\overline{\imath}} + \sqrt{\hbar}\nabla_i$$

act like ladder operators of an N-dimensional harmonic oscillator.

Definition (Toeplitz quantization map)

Let $\mathcal{A}_0$ be the subspace of Schwarz functions in the classical algebra and $\mathcal{K}(\mathcal{H}_\hbar) \subseteq \mathcal{B}(\mathcal{H}_\hbar)$ be the algebra of compact operators. The Toeplitz quantization map

$$T_\hbar : \mathcal{A}_0 \to \mathcal{K}(\mathcal{H}_\hbar)$$

is given by the projector $\Pi_\hbar : \mathrm{L}^2(\mathcal{S}, \mathcal{L}_\hbar) \to \mathcal{H}_\hbar$ as

$$\forall \psi \in \mathcal{H}_\hbar : \qquad T_\hbar(f)\psi = \Pi_\hbar(f\psi).$$

Toeplitz quantization extends to the bounded operators $\mathcal{B}(\mathcal{H}_\hbar)$.

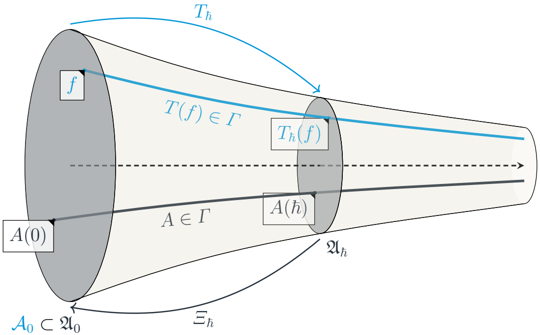
Let $\mu_\hbar$ be a measure such that for compactly supported functions f,

$$\mathrm{Tr}\big(T_\hbar(f)\big) = \int_{\mathcal{S}} f \, \mathrm{d}\mu_\hbar .$$

Definition

The Berezin-Toeplitz dequantization is a family of linear maps $\Xi_\hbar : \mathfrak{A}_\hbar \to \mathfrak{A}_0$ such that for all complex-valued, compactly supported functions $f \in \mathrm{C_c}(\mathcal{S}, \mathbb{C})$ and all operators $A_\hbar \in \mathfrak{A}_\hbar$

$$\mathrm{Tr}\big(A_\hbar T_\hbar(f)\big) = \frac{1}{(2\pi\hbar)^N} \int_{\mathcal{S}} \Xi_\hbar(A_\hbar) f \, \mathrm{dvol}.$$



By construction, this map respects the involution, $\Xi_\hbar(A^*) = \overline{\Xi_\hbar(A)}$, and is normalized.

Lemma

For $\phi \in \mathcal{S}^* = \mathrm{Hom}(\mathcal{S}, \mathbb{R})$, denote the complex components as $\phi_i \in \mathbb{C}$ such that (in the summation convention)

$$\phi(z) = \phi_i z^i + \overline{\phi}_{\bar{\imath}} \overline{z}^{\bar{\imath}}.$$

Let

$$\Phi_\hbar(\phi) := \sqrt{\hbar}\, \delta^{i\bar{\imath}} \left(\phi_i a^+_{\bar{\imath}} + \overline{\phi}_{\bar{\imath}} a^-_i\right).$$

The functions $W_\hbar : \mathcal{S}^* \to \mathcal{B}(\mathcal{H}_\hbar)$ with

$$W_\hbar(\phi) := \exp\big(\mathrm{i}\Phi_\hbar(\phi)\big)$$

fulfill the Weyl relations.

Expansion of the square

$$\begin{aligned}\Phi_\hbar(\phi)^2 &= \hbar\delta^{i\bar{\imath}}\delta^{j\bar{\jmath}}\big(\phi_i\phi_j a^+_{\bar{\imath}} a^+_{\bar{\jmath}} + 2\overline{\phi}_{\bar{\imath}}\phi_j a^-_i a^+_{\bar{\jmath}} \\ &\qquad + \overline{\phi}_{\bar{\imath}}\overline{\phi}_{\bar{\jmath}} a^-_i a^-_j\big) - \hbar|\phi|^2\mathbf{1}, \\ &= \hbar\delta^{i\bar{\imath}}\delta^{j\bar{\jmath}}\big(\phi_i\phi_j a^+_{\bar{\imath}} a^+_{\bar{\jmath}} + 2\phi_i\overline{\phi}_{\bar{\jmath}} a^+_{\bar{\imath}} a^-_j \\ &\qquad + \overline{\phi}_{\bar{\imath}}\overline{\phi}_{\bar{\jmath}} a^-_i a^-_j\big) + \hbar|\phi|^2\mathbf{1},\end{aligned}$$

$$|\phi|^2 := \sum_{i=1}^{N} |\phi_i|^2$$

- the first is anti-normal ordered, compatible with T-quantization
- the second is normal ordered, compatible with Ξ-dequantization

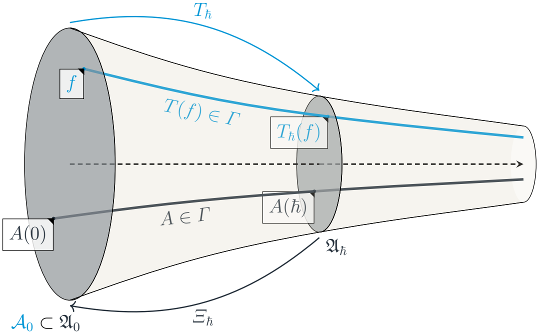
The extra terms of all orders combine to an exponential amplitude factor for quantization

$$W_\hbar(\phi) = \exp\left(\frac{\hbar}{2}|\phi|^2\right) T_\hbar\left(\mathrm{e}^{\mathrm{i}\phi}\right),$$

and dequantization

$$\varXi_\hbar\left(W_\hbar(\phi)\right) = \exp\left(-\frac{\hbar}{2}|\phi|^2\right) \mathrm{e}^{\mathrm{i}\phi}.$$

In the classical limit $\hbar \to 0$, all expressions reduce to $\mathrm{e}^{\mathrm{i}\phi}$.



Definition (Weyl section)

The Weyl section of $\phi \in \mathcal{S}^* = \mathrm{Hom}(\mathcal{S}, \mathbb{R})$ is

$$W(\phi) : \hbar \mapsto \begin{cases} \mathrm{e}^{\mathrm{i}\phi} & \hbar = 0, \\ W_\hbar(\phi) & \hbar > 0. \end{cases}$$

Classical Fourier transform

Recall that any Schwartz function $f \in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S}, \mathbb{C})$ has a Fourier transform

$$\hat{f}(\phi) = \frac{1}{(2\pi)^{2N}} \int_{\mathcal{S}} f(z) \mathrm{e}^{-\mathrm{i}\phi(z)} \, \mathrm{dvol}(z),$$

and an inverse

$$f(z) = \int_{\mathcal{S}^*} \hat{f}(\phi) \mathrm{e}^{\mathrm{i}\phi(z)} \, \mathrm{dvol}^*(\phi).$$

"Quantum" Fourier transform

Let

$$\sqrt{\widehat{b_\hbar}}(\phi) = \frac{1}{(2\pi)^N} \exp\left(-\frac{\hbar}{2}|\phi|^2\right).$$

The Toeplitz operator of f is then

$$\begin{aligned} T_\hbar(f) &= \int_{\mathcal{S}^*} \hat{f}(\phi) T_\hbar\left(\mathrm{e}^{\mathrm{i}\phi}\right) \, \mathrm{dvol}^*(\phi) \\ &= (2\pi)^N \int_{\mathcal{S}^*} \hat{f}(\phi) \sqrt{\widehat{b_\hbar}}(\phi) W_\hbar(\phi) \, \mathrm{dvol}^*(\phi). \end{aligned}$$

For more relations between the Toeplitz and Weyl quantization maps, see [Landsman 1998, ch. II].

Theorem (Sorkin-Johnston state from dequantization)

The linear map $\sigma_\hbar : \mathfrak{A}_\hbar \to \mathbb{C}$ *given by*

$$\sigma_\hbar(A) := \Xi_\hbar(A)(0)$$

is the Sorkin-Johnston state.

Note that

$$\sigma_\hbar\big(W_\hbar(\phi)\big) = \exp\left(-\frac{\hbar}{2}|\phi|^2\right).$$

For any Toeplitz operator $T_\hbar(f) \in \mathfrak{A}_\hbar$ $(f \in \mathcal{A}_0)$:

$$\sigma_\hbar\big(T_\hbar(f)\big) = (\Xi_\hbar \circ T_\hbar)(f)(0).$$

Berezin transform

The Berezin transform

$$(\Xi_\hbar \circ T_\hbar)(f) = b_\hbar \circledast f$$

is a convolution with the kernel

$$b_\hbar(z) := \frac{1}{(2\pi\hbar)^N} \exp\left(-\frac{1}{\hbar}|z|^2\right).$$

So the state of a Toeplitz operator is a smearing with the Berezin kernel

$$\sigma_\hbar\big(T_\hbar(f)\big) = \frac{1}{(2\pi\hbar)^N} \int_{\mathcal{S}} \mathrm{e}^{-\frac{1}{\hbar}|z|^2} f(z)\,\mathrm{dvol}(z).$$

Summary of the construction

Construction of a quantum algebra and a distinguished state on

- a real vector space $\mathcal{S}$
- with an inner product $\langle \cdot, \cdot \rangle$
- and a symplectic form ω such that $\omega(\varphi_1, \varphi_2) = \left\langle \varphi_1, E^{-1} \varphi_2 \right\rangle$.

Generalization to interacting field theory

- The coupling constant λ in an interacting theory is another deformation parameter, giving a perturbation series in λ [DableHeath-Fewster-Rejzner-Woods 2020].
- Then the phase space is no longer a symplectic vector space, but geometric quantization may still be applicable.
- Whenever geometric quantization works, this may give rise to a non-perturbative derivation of a quantum algebra.

Thank you for your interest!

Continuous field of C*-algebras, after [Dixmier 1964, 1977]:

Let I be a topological space and let $(\mathfrak{A}_\hbar)_{\hbar \in I}$ be a family of C*-algebras. A continuous field of C*-algebras is a triple $\big(I, (\mathfrak{A}_\hbar)_{\hbar \in I}, \varGamma\big)$ with vector fields $\varGamma \subseteq \prod_{\hbar \in I} \mathfrak{A}_\hbar$ such that

1. $\varGamma$ is a linear subspace of $\prod_{\hbar \in I} \mathfrak{A}_\hbar$, closed under multiplication and involution,
2. for every $\hbar \in I$ the set $\{A(\hbar) \in \mathfrak{A}_\hbar | A \in \varGamma\}$ is dense in $\mathfrak{A}_\hbar$, and
3. for every element $A \in \varGamma$ the norm function $n_A : I \to \mathbb{R}$ defined by
$$n_A(\hbar) := \|A(\hbar)\|$$
is continuous, $n_A \in \mathrm{C}(I, \mathbb{R})$, as well as
4. if a vector field $A' \in \prod_{\hbar \in I} \mathfrak{A}_\hbar$ fulfills the condition that for all $\hbar \in I$ and for all real constants $\delta > 0$ there exists a neighborhood $N_\hbar \subset I$ of $\hbar$ such that
$$\exists A \in \varGamma : \forall \hbar' \in N_\hbar : \qquad \|A'(\hbar') - A(\hbar')\| \leq \delta,$$
then A' is also a section, $A' \in \varGamma$.

The elements of $\varGamma$ are called (continuous) sections of the field.