Local symmetries in partially ordered sets and causal sets

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Abstract

Starting from a graph of sequential growth, we group partially ordered sets (posets) into classes and characterise locally repeating structures (local symmetries) in posets. The aim is to find subsets of posets that can be enumerated using combinatorial arguments more easily. The study of local symmetries applied to causal sets may also help to distinguish between generic posets and those that are suited as discrete representations of spacetime manifolds, as discussed in this talk.
Alongside, we will see some tools to type-set posets with LaTeX and compose them in a visual editor. I made these tools available via CTAN (to be included in LaTeX distributions and on Overleaf) and on `https://c-minz.github.io/projects/poset_diagrams`.

Tools for causal sets and research on local symmetries

1. LaTeX-package 'causets' to draw Hasse diagrams (of causal sets and partially ordered sets in general), → [CTAN 2020] *ctan.org/pkg/causets*,
2. Online tool to help finding the LaTeX-macros, → [M 2024] *c-minz.github.io*,
3. Preprint "Local symmetries in partially ordered sets". → [M 2024] *arXiv:2406.14533*.

1. The enumeration problem and sequential growth
2. Local symmetries in posets
3. Posets of regular geometric polytopes
4. The enumeration of posets using local symmetries
5. Local symmetries in causets

The enumeration problem of posets

The OEIS lists the number of all posets (with n unlabelled elements) as sequence A000112:

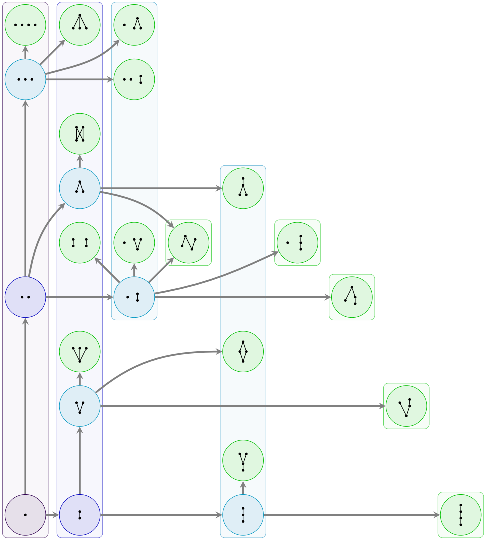
$$1, 1, 2, 5, 16, 63, 318, 2045, 16999, \dots,$$

which is so far only known up to $n = 16$.

Is it possible to find a closed expression for this sequence, at least for certain subclasses of posets?

Sequential growth

Growth rule used here (arrows): any newly added element has to appear in the top-layer



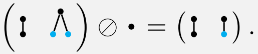
Singleton-symmetric elements

Let P be a poset. Two elements $a, b \in P$ are *singleton-symmetric* if they precede and succeed the same elements (the have the same link past and link future),

$$
\begin{aligned}
&L^{\pm}(a) = L^{\pm}(b) \\
\Leftrightarrow\ &J_*^{\pm}(a) = J_*^{\pm}(b)\,.
\end{aligned}
$$

- $\Rightarrow$ "Singleton-symmetric" is an equivalence relation.
- $\Rightarrow$ Taking the quotient of a poset P by this symmetry yields a *retract* denoted by $P \oslash \bullet$.

Example (2-chain and wedge)



LaTeX code

```
\usepackage{causets}
...
\begin{equation*}
  ( \pcauset{4,3,5,1,2} )
  ( \pcauset{4/cyan,3/cyan,5,1,2} )
  \oslash \pcauset{1}
  = ( \pcauset{3/cyan,4,1,2} )
\end{equation*}
```

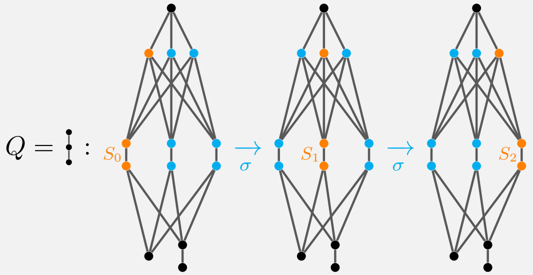
Notation

- poset P, finite poset Q, integer $r \geq 2$
- automorphism $\sigma \in \mathrm{Aut}(P)$
- $\Sigma(\sigma)$ is the set of elements that are not fixed under σ, so $P \setminus \Sigma(\sigma)$ are fixed

Automorphisms as symmetry generators

A (Q, r)-generator is an automorphism σ such that there exists a sequence of r (smallest, maximally ordered) subsets $S_i \subset \Sigma(\sigma)$ with $S_i \cong Q$, $\sigma(S_i) = S_{i+1 \mod r}$ $(0 \leq i < r)$ that cover $\Sigma(\sigma)$.

Example (A local symmetry that is composite)



LaTeX code

```
\tikzcausetsset{large, gray colors,
S/.style={cyan}, S1/.style=orange} ...
\pcauset{2,3,1,8/S,9/S,6/S1,7/{label=$S_1$,
labels=S1, S1},4/S,5/S,12/S,11/S1,10/S,13}
```

(Q, r)-symmetric elements

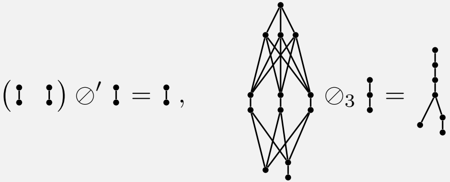
Let σ be a (Q, r)-generator in a poset P.

- Two elements $a, b \in P$ are $(Q, r, 0)$-*symmetric* $(a \sim_0 b)$ if $a = b$.
- They are $(Q, r, 1)$-*symmetric* $(a \sim_1 b)$ if $a \in A \subset P$ and $b \in B \subset P$ such that A and B are (Q, r)-symmetric and $b = \sigma^q(a)$ for some $1 \leq q < r$.
- They are (Q, r, n)-*symmetric* $(a \sim_n b)$ if $a \not\sim_j b$ for any $j < n$, but $\exists c \in P$ and $j < n$ such that $a \sim_j c$ and $c \sim_j b$.

If $\exists n \in \mathbb{N}_0$ such that $a \sim_n b$, for short, we say a is (Q, r)-*symmetric* to b.

Quotient by all (Q, r)-symmetries gives a *retract* $P \oslash_r Q$ (and we drop the index if $r = 2$).

Example (Retraction of local symmetries)

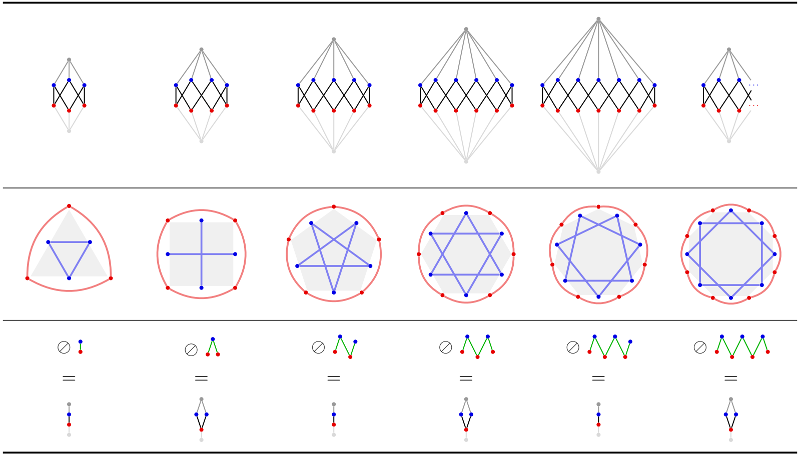


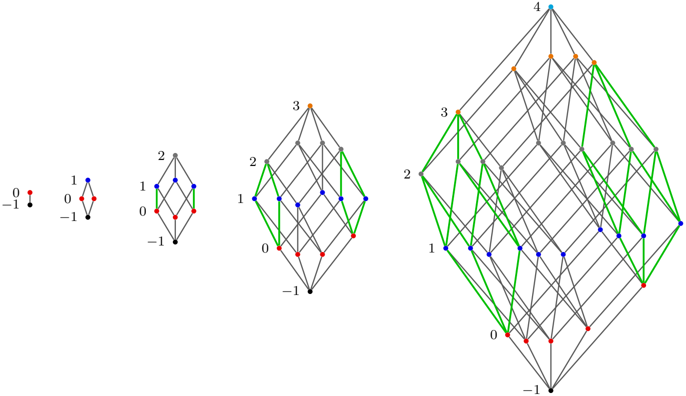
LaTeX code

```
( \pcauset{3,4,1,2} ) \oslash \pcauset{1,2}
= \pcauset{1,2}
...
\pcauset{2,3,1,8,9,6,7,4,5,12,11,10,13}
\oslash_3 \pcauset{1,2,3}
= \pcauset{2,3,1,4,5,6,7}
```

Link to the PrOSET editor at:
`https://c-minz.github.io/projects/poset_diagrams`

Posets of polygons (cycles of the 2-chain, with dihedral symmetry)





Theorem (Simplices)

The d-simplex is $(d-2)$-simplex-retractable to the $(d+2)$-chain.

Theorem (Preservation of layers)

For any (Q, r)-symmetric poset P, the symmetry quotient $P/(Q, r)$ preserves layers.

Symmetry properties

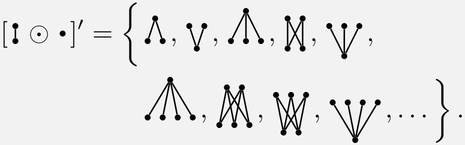
- A poset P is *locally* (Q, r)*-symmetric* and (Q, r)*-retractable* to the poset $\tilde{P}$ if $\tilde{P} = P \oslash_r Q \neq P$.
- It is *locally symmetric* if there is some pair (Q, r) such that P is locally (Q, r)-symmetric.
- It is *retractable* to the poset $\tilde{P}$ (the retract of P) if there exist some sequence of (Q_i, r_i)-symmetries such that $\tilde{P} = P \oslash_{r_1} Q_1 \oslash_{r_2} Q_2 \oslash_{r_3} \ldots \neq P$.
- It is *locally unsymmetric* if it is not locally symmetric.

Classes of symmetry extensions

All posets that are (Q, r)-retractable to some poset R form a class of *symmetry extensions*

$$[R \odot_r Q] := \{P \in \mathfrak{P} \mid P \oslash_r Q = R \neq P\} .$$

Example (Posets of complete bipartite graphs)



Counting function

Let $\mathfrak{P}$ be the set of all posets (with unlabelled elements). On its power set, we have the counting function

$$c_n : \mathcal{P}(\mathfrak{P}) \to \mathbb{N}_0 \,, \qquad \mathfrak{C} \mapsto \Big| \big\{ P \in \mathfrak{C} \;\big|\; |P| = n \big\} \Big| \,.$$

For any locally unsymmetric poset R with cardinality $r := |R|$, if $n > r$, then

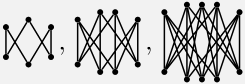
$$c_n[R \odot \bullet]' = \binom{n-1}{r-1} \,.$$

Example (Posets of complete bipartite graphs)

$$c_n[\mathrm{I} \odot \bullet]' = n - 1 \text{ if } n > 2 \,.$$

Example (Repeated 2-chains and crowns)

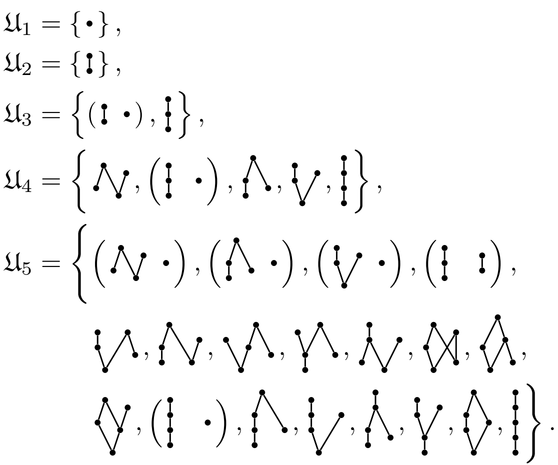
$$[\mathrm{I} \odot \mathrm{I}]' = \Big\{ (\mathrm{I} \;\; \mathrm{I}), (\mathrm{I} \;\; \mathrm{I} \;\; \mathrm{I}), (\mathrm{I} \;\; \mathrm{I} \;\; \mathrm{I} \;\; \mathrm{I}), \ldots,$$

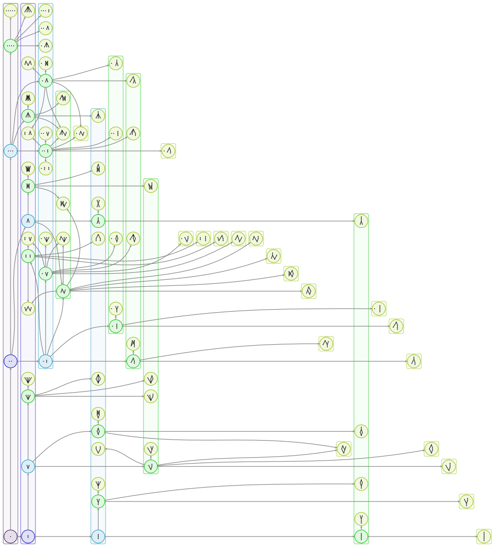


$$, \ldots \Big\}$$

$$c_n[\mathrm{I} \odot \bullet]' = \begin{cases} 2 & \text{if } n \text{ is even and } n \geq 6, \\ 1 & \text{if } n = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Locally unsymmetric posets





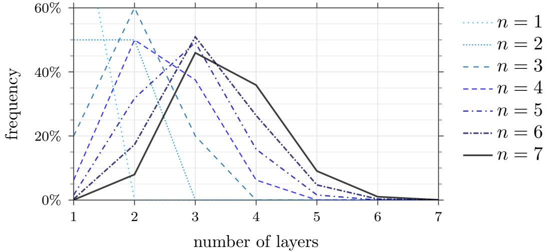
From all posets $\mathfrak{P}$, we count those with cardinality n (rows) and layers l (columns),

$$p(n,l) := \left| \left\{ P \in \mathfrak{P} \;\middle|\; |P| = n, \ell(P) = l \right\} \right|,$$

$$p_n := \left| \left\{ P \in \mathfrak{P} \;\middle|\; |P| = n \right\} \right| = \sum_{l=1}^{n} p(n,l)\,,$$

and the frequency is $p(n,l)/p_n$.

	1	2	3	4	5	6	7	p_n
1	1							1
2	1	1						2
3	1	3	1					5
4	1	8	6	1				16
5	1	20	31	10	1			63
6	1	55	162	84	15	1		318
7	1	163	940	734	185	21	1	2045



Theorem

$$p(n,n-1) = \frac{1}{2}n(n-1)\,,$$

$$p(n,n-2) = \frac{1}{6}(n-2)\left(n^3 - 2n^2 - 5n + 12\right).$$

Similar for the set of all locally unsymmetric posets $\mathfrak{U}$ we get the counts $u(n, l)$ and u_n.

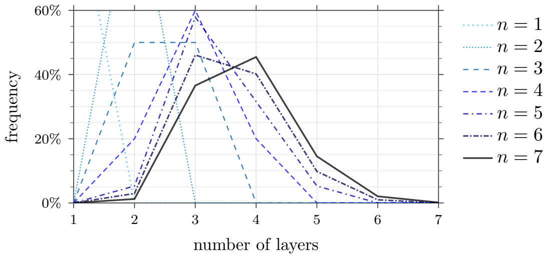
Theorem

$$u(n, n-1) = \frac{1}{2}(n-1)(n-2)\,,$$
$$u(n, n-2) = \frac{1}{6}(n-3)\left(n^3 - 4n^2 + 2n - 2\right).$$

Comparison with all posets

$$\frac{u(n, n-1)}{p(n, n-1)} = 1 - \frac{2}{n}\,,$$
$$\frac{u(n, n-2)}{p(n, n-2)} = 1 - \frac{3}{n} + \frac{3}{n^2} + \mathcal{O}\left(n^{-3}\right).$$

	1	2	3	4	5	6	7	u_n
1	1							1
2	0	1						1
3	0	1	1					2
4	0	1	3	1				5
5	0	1	11	6	1			19
6	0	3	47	41	10	1		102
7	0	9	267	332	106	15	1	730

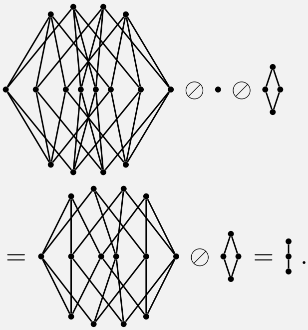


Open problems

- Computing the numbers $p(n, l)$ and $u(n, l)$ for $l = n - 3$, $l = n - 4$ and so on becomes increasingly more complicated.
- Kleitman–Rothschild orders, which are sufficient to estimate the large n behaviour [Kleitman–Rothschild 1975], may not contain local symmetries with high enough probability.
- At least, is it possible to enumerate all two-layer posets?

Example (Some Kleitman–Rothschild orders have local symmetries)

Fig. 1 from [Carlip–Carlip–Surya 2023] is retractable to the $(0, 1, 2)$-faces subset of the 3-simplex and to the 3-chain,



Stability of local asymmetry

Let $k \in \mathbb{N}_0$.

- A poset P is *k-stable locally unsymmetric* if, for every subset $S \subseteq P$ that has cardinality $0 \leq |S| \leq k$, the poset $P \setminus S$ is locally unsymmetric.
- A poset P is *total locally unsymmetric* if $P \setminus S$ is k-stable locally unsymmetric for every finite $k < |P|$.

Example

Any chain posets (total order) is total locally unsymmetric.

	1	2	3	4	5	6	7	s_n
1	1							1
2	0	1						1
3	0	0	1					1
4	0	0	1	1				2
5	0	0	0	3	1			4
6	0	0	2	8	6	1		17
7	0	0	4	37	36	10	1	88

Enumeration for 1-stable locally unsymmetric posets by cardinality (rows) and layer (columns).

- Almost all posets with two or three layers and cardinality $n \leq 7$ are not even 1-stable locally unsymmetric.
- Local asymmetries in posets tend to have a higher stability the more layers there are.

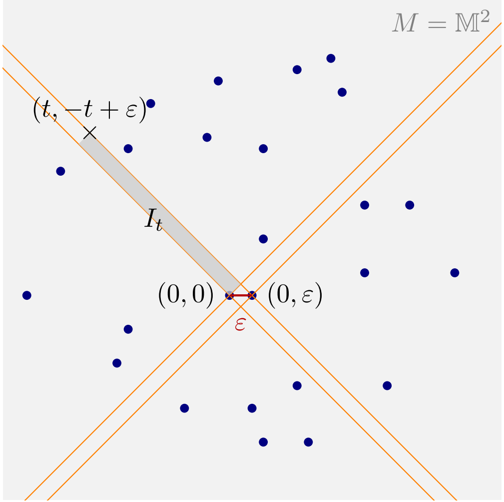
Theorem

A sprinkle in Minkowski spacetime $\mathbb{M}^{1+d}$ *is total locally unsymmetric with probability 1.*

Proof: Let S be a random sprinkle in Minkowski spacetime, take two separated elements that are at the top layer of any assumed local symmetry (after removing a finite number of elements from their future). The probability for a region I_t to contain n elements is

$$\Pr\Big(|\mathsf{S} \cap I_t| = n\Big) = \frac{\rho^n \nu(I_t)^n}{n!} \mathrm{e}^{-\rho\nu(I_t)} .$$

To be total locally unsymmetric I_t could at most contain finitely many elements, even if t is arbitrarily large ($t \to \infty$), so this probability vanishes no matter how small $\varepsilon > 0$ is. □



Summary

- (Infinite) sprinkles do not have local symmetries (with probability 1).
- What are the consequences to the causal set action when we consider only locally unsymmetric causal sets?
- Are local symmetries relevant or even necessary to model the very early universe in causal set theory?

LaTeX-package 'causets'

It is part of full LaTeX installations, for example, it is available on Overleaf. Just load the package with `\usepackage{causets}`.

Thank you for your interest!