Modular Hamiltonians of the massless and massive scalar field on the half line

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Linear quantum field theory on the half space

We consider the scalar field, which fulfills the field equations with spatial boundary at $x = 0$,

$$(\partial_t^2 + D)\psi = 0\,, \quad D \coloneqq -\partial_x^2 + m^2\,,$$
$$\lim_{x\to 0^+}\left(\psi' - \eta\psi\right) = 0\,.$$

We want to compute the modular Hamiltonian $H = -\log \Delta$ corresponding to the vacuum state and a subspace region.

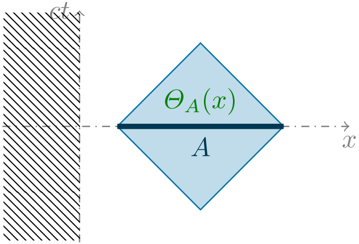
One-particle structure

Real, separable Hilbert space $\mathcal{H}_\mathrm{r} \coloneqq \mathrm{L}^2_\mathbb{R}(\mathbb{R}^{d-1})$ and subspace $\mathcal{H}_{\mathrm{r},A} \coloneqq \mathrm{L}^2_\mathbb{R}(A)$ for two pieces of initial data on the time-0 hypersurface.

Together they form a complex Hilbert space $\mathcal{H} = \mathcal{H}_\mathrm{r}^{1/4} \oplus \mathcal{H}_\mathrm{r}^{-1/4}$ with **complex structure**

$$I = \begin{pmatrix} 0 & D^{-\frac{1}{2}} \\ -D^{+\frac{1}{2}} & 0 \end{pmatrix},$$

and a *standard* subspace $\mathcal{H}_A \subset \mathcal{H}$. with **projector** $P = \Theta_{+\frac{1}{4},A} \oplus \Theta_{-\frac{1}{4},A}$, [Bostelmann–Cadamuro–M. '23]



Staying at the one-particle level

For all self-adjoint operators $F = F^*$, $G = G^* \in \mathcal{A}(\mathcal{O})$ acting on the vacuum state Ω

$$\hat{T} : (F + iG)\Omega \mapsto (F - iG)\Omega$$

which corresponds to the complex conjugation at the one-particle level ($f, g \in \mathcal{H}_A$)

$$f + Ig \mapsto f - Ig \,,$$

with the Tomita operator $T = J\Delta^{1/2}$ as the closure.
On Fock space, the modular operator is the second quantization $\hat{\Delta} = \Gamma(\Delta)$
for bosons [Eckmann–Osterwalder 1973] and
for fermions [Foit 1983]; see also [Longo 2019].

From the projectors to the modular operator

The one-particle Tomita operator $T = J\Delta^{1/2}$ and the cutting projection P are related [Figliolini–Guido 1994, Longo 2022],

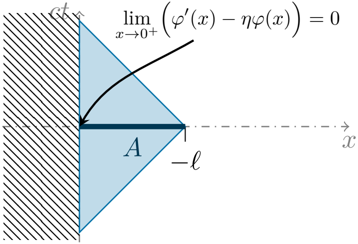
$$-I \log \Delta = 2I \operatorname{arcoth}(P - IPI - 1) \,.$$

$$-I \log \Delta = \begin{pmatrix} 0 & M_- \\ -M_+ & 0 \end{pmatrix} ,$$

$$M_\pm = 2D^{\pm\frac{1}{4}} \operatorname{arcoth}(Z) D^{\pm\frac{1}{4}}$$

$$Z = D^{+\frac{1}{4}} \Theta_{+\frac{1}{4},A} D^{-\frac{1}{4}} + D^{-\frac{1}{4}} \Theta_{-\frac{1}{4},A} D^{+\frac{1}{4}} - 1 \,.$$

Region over an adjacent interval [Cardy–Tonni 2016]

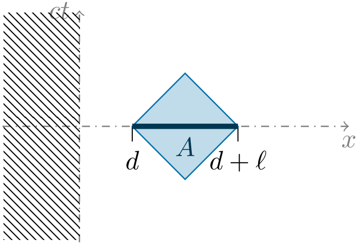


Boundary conformal field theory (massless boson) for a Neumann or Dirichlet boundary condition:

$$M_-(x, y) = 2\pi \frac{\ell^2 - x^2}{2\ell} \delta(x - y) \,,$$

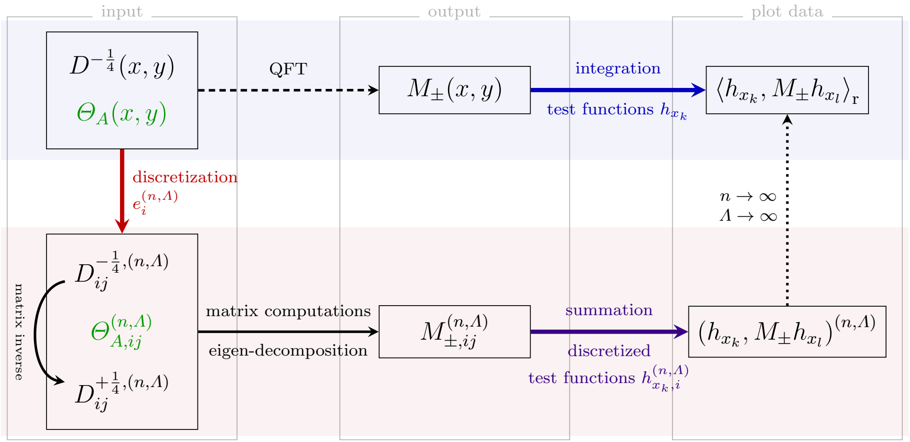
(shown with a magenta line in later plots).

Region over a separated interval

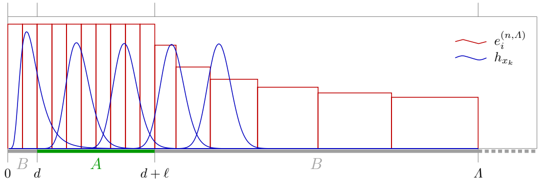


In the bosonic case, there is no known solution for this region in the massless and massive regimes yet.

Scheme of the numerical approach



Functions for discretization and smearing

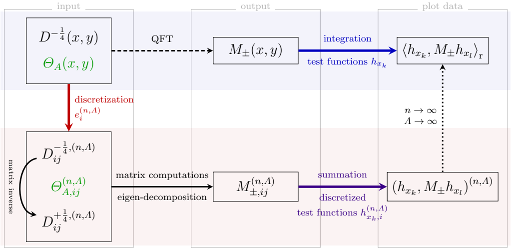


$$e_i^{(n,\Lambda)}(x) := \frac{\Theta(x-a_i)\Theta(b_i-x)}{\sqrt{b_i-a_i}}\,,$$

$$O_{ij}^{(n,\Lambda)} := \left\langle e_i^{(n,\Lambda)},\, O e_j^{(n,\Lambda)} \right\rangle_{\mathrm{r}}\,,$$

$$h_{x_k}(x) := \mathrm{L}^2\text{-normalized log-Gaussian}\,,$$

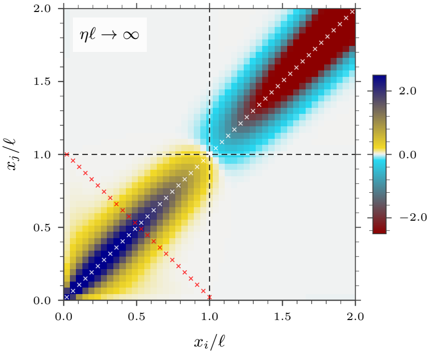
$$h_{x_k,i}^{(n,\Lambda)} := \left\langle e_i^{(n,\Lambda)},\, h_{x_k} \right\rangle_{\mathrm{r}}\,,$$

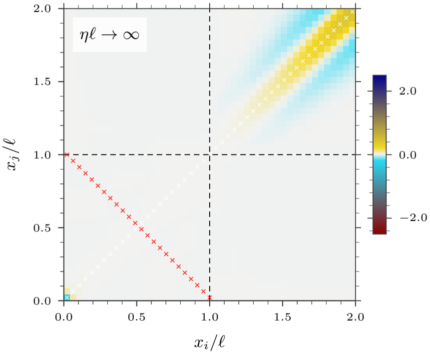


$$\langle h_{x_k},\, M_\pm h_{x_l}\rangle_{\mathrm{r}}\,,$$

$$\sum_{i,j=0}^{n-1} h_{x_k,i}^{(n,\Lambda)} M_{\pm,ij}^{(n,\Lambda)} h_{x_l,j}^{(n,\Lambda)} =: (h_{x_k},\, M_\pm h_{x_l})^{(n,\Lambda)}\,.$$

A scalar boson on the adjacent interval, massless regime





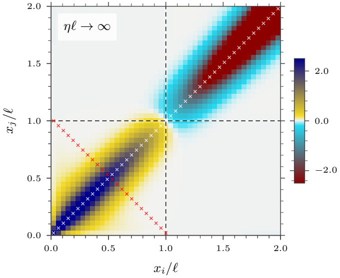
Smeared numeric results

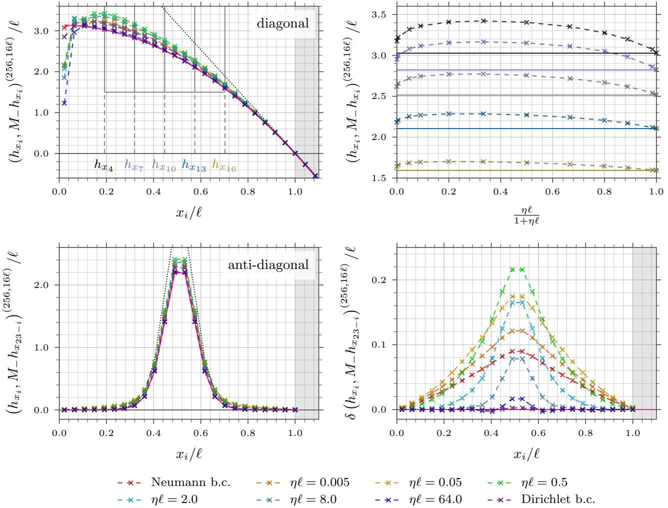
$$(h_{x_k}, M_- h_{x_l})^{(n,\Lambda)} := \sum_{i,j} h^{(n,\Lambda)}_{x_k,i} M^{(n,\Lambda)}_{\pm,ij} h^{(n,\Lambda)}_{x_l,j}$$

Smeared difference between the numeric results and the analytic BCFT reference

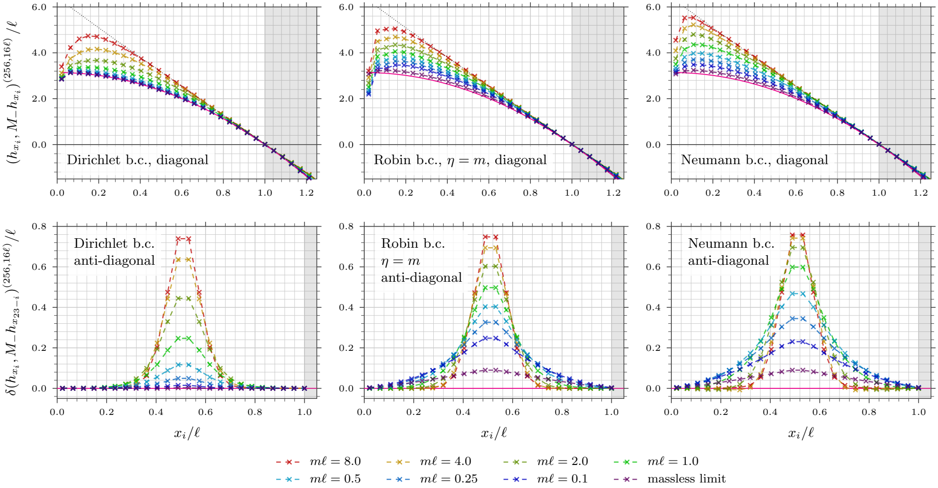
$$(h_{x_k}, M_- h_{x_l})^{(n,\Lambda)} - \langle h_{x_k},\, M_-^{\mathrm{BCFT}} h_{x_l} \rangle_{\mathrm{r}}$$

Dependence on the boundary parameter η in the massless regime

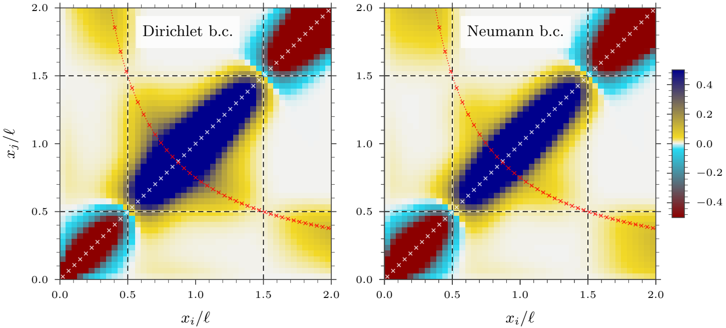




Dependence on the mass parameter m



Plotting curves



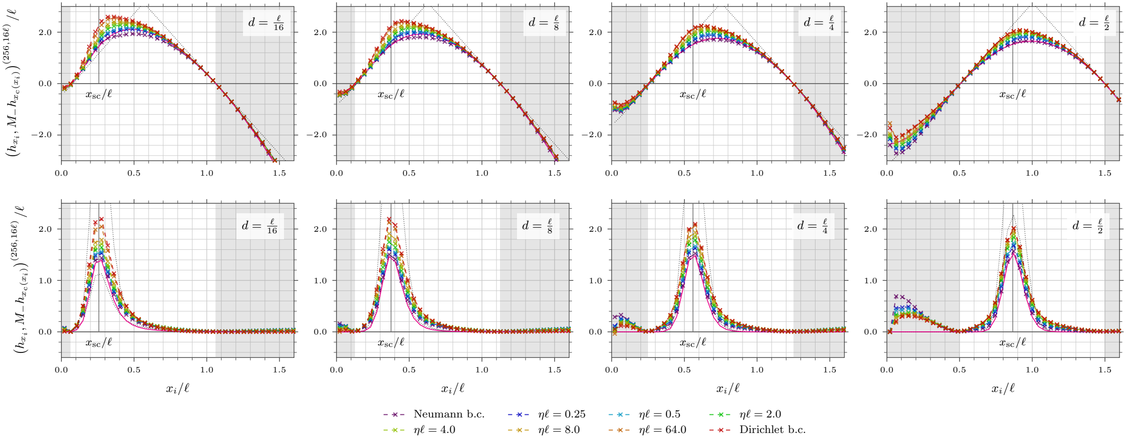
Main diagonal (white crosses) and conjugate curve (red crosses),

$$x_{\mathrm{c}}(x) := \frac{d(d+\ell)}{x} .$$

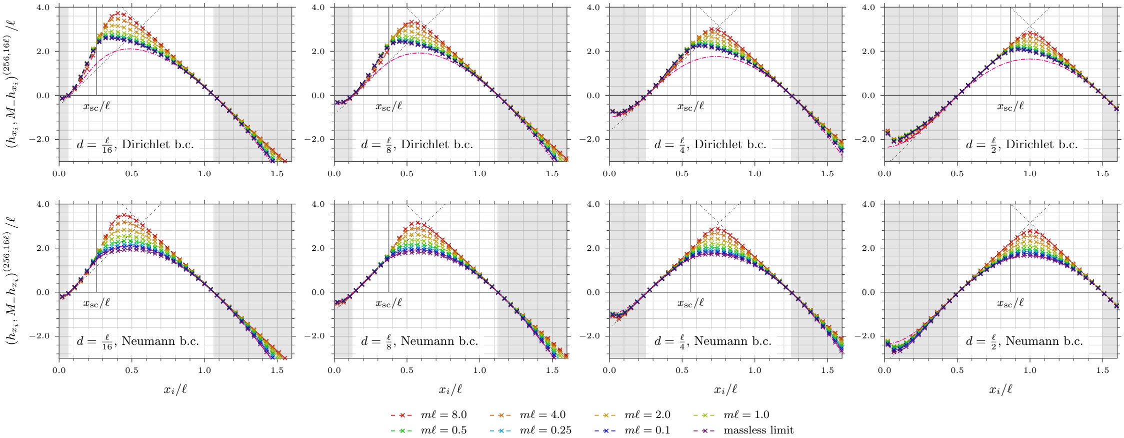
For the separated interval, there is no known bosonic reference, so we plot the local part of the corresponding fermionic BCFT reference [Mintchev–Tonni 2021] (dash dot magenta line below)

$$M_{-}^{\mathrm{loc}}(x,y) = \frac{\pi}{\ell} \frac{\left[x^2 - d^2\right]\left[(d+\ell)^2 - x^2\right]}{d(d+\ell) + x^2} \delta(x-y) .$$

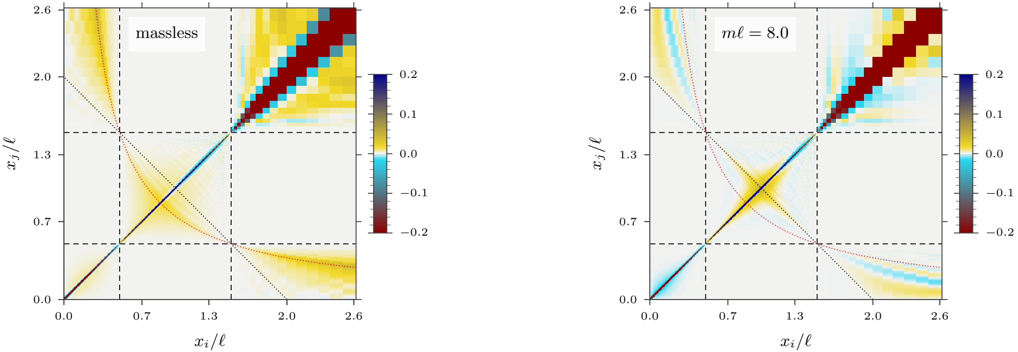
Dependence on the boundary parameter η and the separation distance d



Dependence on the mass parameter m and the separation distance d



Dependence on the mass parameter m (here with a Dirichlet boundary)



$$x_c(x) := \frac{d(d+\ell)}{x}$$ (dotted red)

$$y(x) := \ell + 2d - x$$ (dotted black)

Discretized numeric results $M_-^{(n,\Lambda)}$ (without smearing).
Non-diagonal contributions are mostly concentrated along and between these two curves.

Summary

- The modular Hamiltonian $-\log \Delta$ is can be used to compute relative entropies in QFT, for example, between the Fock vaccum ω and a coherent excitation described by the Weyl operator $W(f)$ for any vector f in the bosonic Hilbert space $\mathcal{H}$,

$$\omega_f(\hat{a}) := \omega\big(W(f)^* \hat{a} W(f)\big)\,,$$
$$S\left(\omega_f \| \omega\right) = \Big\langle f,\, \big(\Theta_{-\frac{1}{4},A} M_+ \oplus \Theta_{\frac{1}{4},A} M_-\big) f \Big\rangle_{\frac{1}{4} \oplus -\frac{1}{4}}\,.$$

- We can approximate the modular Hamiltonian numerically.

Challenges

- We want to consider a similar setup in higher dimensions (where we have a defect at the coordinate origin).
- Is it possible to generalize the approach to curved spacetimes?
- And how can we treat other states or non-linear fields?

Thank you for your interest.