From a spacetime to sprinkled causal sets, and back?

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The path to quantum gravity with causal sets (Manchester)
8 Sep. 2026

Mathematical review of the sprinkling process

- A mathematical review of the probability space for sprinkling on manifolds is in Sec. IV.
- Some investigations of geodesics in sprinkled causets (Sec. 3.4) and the combinatorial method for 2-orders (Sec. 4).
- Computations of the probabilities for small sprinkled causet in differently shaped regions of Minkowski spacetime in $(1+1)$ through $(1+3)$ dimensions.

[Fewster–Hawkins–M.–Rejzner 2021]
doi:10.1103/PhysRevD.103.086020

[M. 2021]
etheses.whiterose.ac.uk/29866 (PhD thesis)

[M. 2020]
(unpublished part of my PhD projects)

Guiding questions for this talk

1. What are suitable properties to define “manifold-like” causets?
2. What are the most likely causets originating from a sprinkling process?
3. How do sprinkled causets (Poisson measure) compare to generic causets (uniform measure)?

Properties of causets

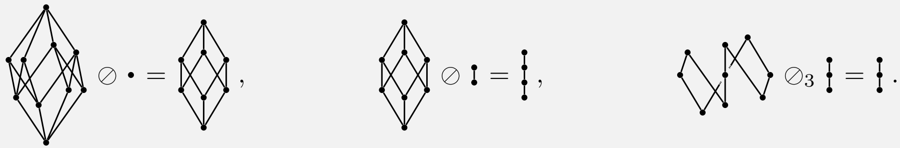
1. Expected length of the longest chain in a causal interval (number of layers)
2. Local symmetries (finite automorphisms)

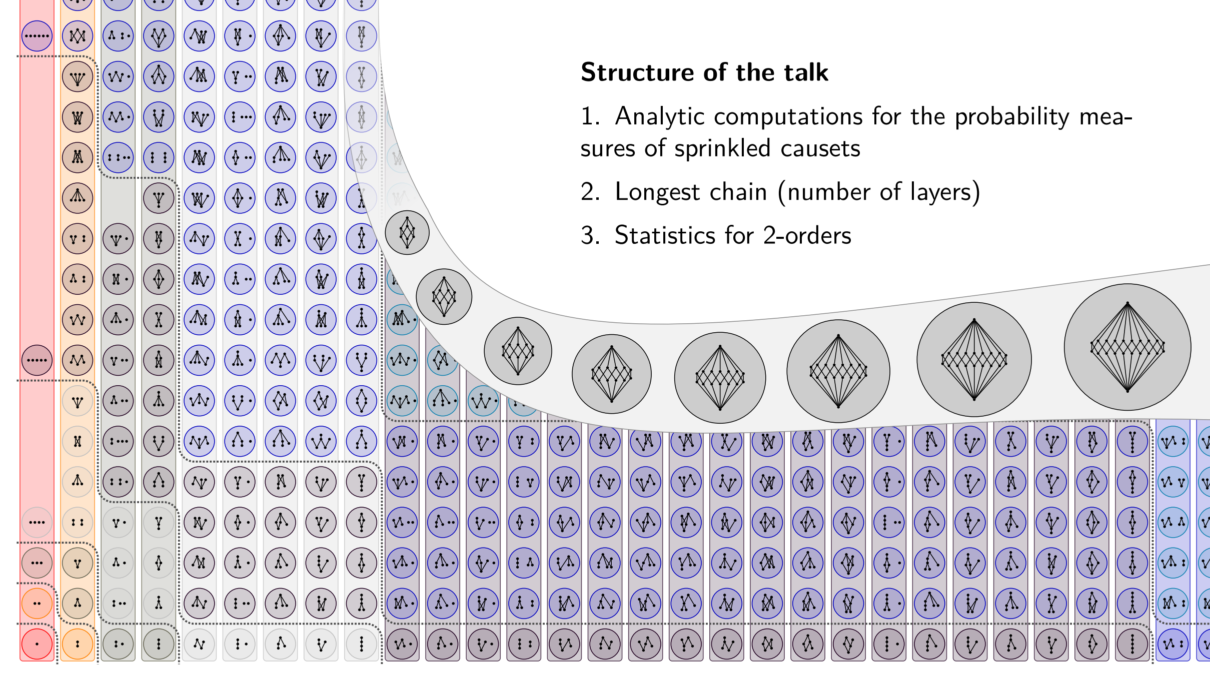
- Definition of local symmetries

[M. 2024]
arXiv:2406.14533

While infinite sprinkles in Minkowski spacetime are (total) locally unsymmetric with probability 1, this is certainly not true for finite causal intervals in sprinkles.

Example (Examples of symmetry retractions by different local (prime) symmetries)





Structure of the talk

1. Analytic computations for the probability measures of sprinkled causets
2. Longest chain (number of layers)
3. Statistics for 2-orders

We consider a double cone region $U \subset M$ of a spacetime manifold M with the configuration spaces [Fewster–Hawkins–M–Rejzner 2021]

$$Q_{U,n} \coloneqq \left\{ S \subset U \;\middle|\; |S| = n \right\},$$
$$Q^{\mathrm{lbl}}_{U,n} \coloneqq \left\{ (x_1, x_2, \dots, x_n) \in U^n \;\middle|\; \forall i, j \in [1, n] : x_i \neq x_j \right\}$$

and the "label forgetting" function $\varUpsilon_{U,n} : Q^{\mathrm{lbl}}_{U,n} \to Q_{U,n}$ maps $n!$ tuples to one n-element set.

The probability measure follows from the volume measure ν on M,

$$\mu_U(B_n) = \mathrm{e}^{-\rho\nu(U)} \frac{\rho^n}{n!} \nu^n\left(\varUpsilon^{-1}_{U,n}(B_n)\right),$$

for any measurable subset $B_n \subset Q_{U,n}$.

For a causet C, we have an equivalence class $[C]_U$ that contains all sprinkles S in U that are order-isomorphic to C, $S \sim C$. The probability for a random sprinkle S to be isomorphic to C is then

$$\mathbb{P}_U(C) = \mathbb{P}_U(S \sim C) \coloneqq \mu_U([C]_U).$$

Let n be the cardinality of the causet C, $n = |C|$.
The probability splits into a numerical Poisson factor and a normalised conditional probability

$$\mathbb{P}_U(C) = \mathbb{P}_U\big(|S| = n\big)\mathbb{P}_U\big(S \sim C \;\big|\; |S| = n\big)\,,$$
$$\mathbb{P}_U\big(|S| = n\big) = \mathrm{e}^{-\rho\nu(U)}\,\frac{\big(\rho\nu(U)\big)^n}{n!}\,,$$
$$\mathbb{P}_{U,n}(C) := \mathbb{P}_U\big(S \sim C \;\big|\; |S| = n\big)\,.$$

For the computation of the probabilities $\mathbb{P}_{U,n}(C)$, we consider some labelling of the elements, where each labelling has the same probability $\mathbb{P}^{\mathrm{lbl}}_{U,n}(C)$. With the number of distinct labellings $\mathrm{lbl}(C)$ using integers $[1, n]$, we have

$$\mathbb{P}_{U,n}(C) = \mathrm{lbl}(C)\mathbb{P}^{\mathrm{lbl}}_{U,n}(C)\,.$$

If C is large (or infinite), we consider only small (and finite) causal intervals. The respective sub-sprinkle is a sprinkle on the corresponding double cone region, again following a Poisson process.

So we consider double cone regions U in Minkowski spacetime of spacetime dimension d and denote the probabilities with $\mathbb{P}_{d,n}(C)$.

Example (The vee causet)

For the vee causet $\vee$, the equivalence class is

$$[\vee]_U = \big\{\{a_1, a_2, a_3\} \subset U \;\big|\; a_1 < a_2, a_1 < a_3\big\}\,.$$

So the pre-image of the label-forgetting map is

$$\Upsilon^{-1}_{U,3}\Big([\vee]_U\Big) = \big\{(x_1, x_2, x_3) \in U^3 \;\big|\; \{x_1 < x_2, x_1 < x_3\} \text{ or } \{x_2 < x_1, x_2 < x_3\} \text{ or } \{x_3 < x_1, x_3 < x_2\}\big\}\,.$$

Since each labelling has the same probability, we may choose an order-preserving labelling for the computation, while $\mathrm{lbl}(\vee) = 3$.

$$\mathbb{P}_{d,3}(\vee) = \frac{\mathrm{lbl}(\vee)}{\big(\nu(U)\big)^3} \int_U \int_{U \setminus J(x_3)} \int_{U \cap J^-(x_2) \cap J^-(x_3)} \mathrm{d}\nu(x_1)\, \mathrm{d}\nu(x_2)\, \mathrm{d}\nu(x_3)\,.$$

Standard notation: $J^-(x_i)$ is the past of x_i, and $J(x_i)$ is the past and future of x_i in M.

The volume $V_d = \nu(U)$ and proper time length τ of double cone regions have a known relation due to the spherically symmetry in spatial dimensions

$$V_d(\tau) = \int_{t=-\frac{\tau}{2}}^{\frac{\tau}{2}} \int_{r=0}^{\frac{\tau}{2}-|t|} S_{d-2} r^{d-2}\,\mathrm{d}r\,\mathrm{d}t = \frac{\sigma_d \tau^d}{\alpha_d} \int_{t=-\frac{1}{2}}^{\frac{1}{2}} \int_{r=0}^{\frac{1}{2}-|t|} r^{d-2}\,\mathrm{d}r\,\mathrm{d}t = \frac{\tau^d}{\alpha_d}\,,$$

where the S_{d-2} is the volume of the unit sphere, α_d is the resulting causal volume factor, and $\sigma_d = 2^{d-1}(d-1)d$ is their product.

The trivial probability for a 1-element sprinkle to be isomorphic to the singleton is

$$\mathbb{P}_{d,1}(\bullet) = \alpha_d V_d(1) = 1\,.$$

The probabilities for some causets can be computed recursively with the integrals

$$\Lambda_d(n-1) = \sigma_d \int_{t=-\frac{1}{2}}^{\frac{1}{2}} \int_{r=0}^{\frac{1}{2}-|t|} \left(\left(\frac{1}{2}+t\right)^2 - r^2\right)^{\frac{d(n-1)}{2}} r^{d-2}\,\mathrm{d}r\,\mathrm{d}t\,.$$

So the probability for a series composition ($\vee$) of a causet C with a singleton (and cardinality $n = |C|$) is

$$\mathbb{P}^{\mathrm{lbl}}_{d,n+1}(C \vee \bullet) = \Lambda_d(n)\mathbb{P}^{\mathrm{lbl}}_{d,n}(C)\,, \qquad \mathbb{P}_{d,n+1}(C \vee \bullet) = (n+1)\Lambda_d(n)\mathbb{P}_{d,n}(C)\,.$$

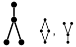
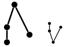
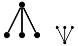
$\mathbb{P}_{d,n}$												$\sum$ other
unif.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{7}{16}$
2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{12}{24}$
3	$\frac{8}{35}$	$\frac{27}{35}$	$\frac{8}{350}$	$\frac{27}{350}$	$\frac{108}{350}$	$\frac{180}{350}$	$\frac{32}{25025}$	$\frac{108}{25025}$	$\frac{432}{25025}$	$\frac{648}{25025}$	$\frac{720}{25025}$	$\frac{21717}{25025}$
4	$\frac{1}{10}$	$\frac{9}{10}$	$\frac{1}{350}$	$\frac{9}{350}$	$\frac{66}{350}$	$\frac{265}{350}$	$\frac{1}{29400}$	$\frac{9}{29400}$	$\frac{66}{29400}$	$\frac{146}{29400}$	$\frac{265}{29400}$	$\frac{28564}{29400}$
5	$\frac{128}{3003}$	$\frac{2875}{3003}$	$\frac{128}{378378}$	$\frac{2875}{378378}$	$\frac{36500}{378378}$	$\frac{336000}{378378}$	$\frac{8192}{9838395567}$	$\frac{184000}{9838395567}$	$\frac{2336000}{9838395567}$	$\frac{7504000}{9838395567}$	$\frac{21504000}{9838395567}$	$\frac{9782651375}{9838395567}$
6	$\frac{1}{56}$	$\frac{55}{56}$	$\frac{1}{25872}$	$\frac{55}{25872}$	$\frac{1163}{25872}$	$\frac{24598}{25872}$	$\frac{1}{51795744}$	$\frac{55}{51795744}$	$\frac{1163}{51795744}$	$\frac{5348}{51795744}$	$\frac{24598}{51795744}$	$\frac{51738708}{51795744}$

Table: Analytic results for the conditional probabilities for sprinkles in double cone (diamond) regions

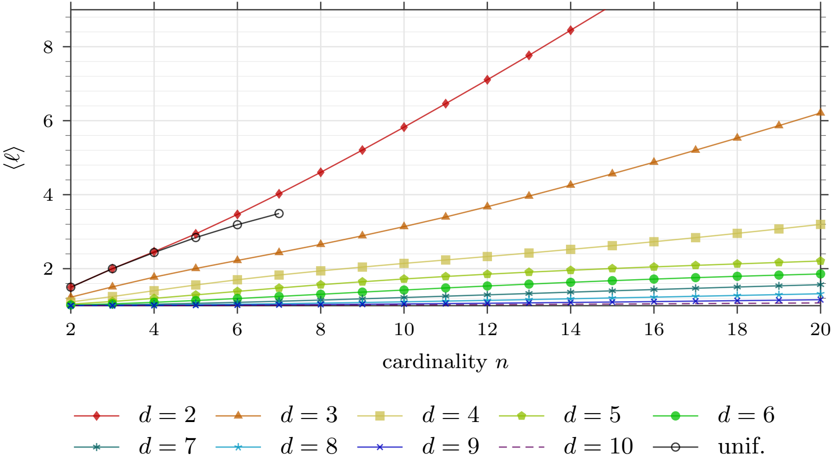


Figure: Expected length of the longest chain (expected number of layers for different sprinkle cardinalities). Each sprinkling data point has been computed as average over 10^6 sprinkles.

We normalise by the d-th root of the cardinality analogous to the ratio, which defines the asymptotic constants for the longest path length

$$m_d \coloneqq \lim_{\rho\to\infty} \frac{\langle \ell \rangle}{\big(\rho\nu(U)\big)^{1/d}} \,.$$

In [Brightwell–Gregory 1991], they have proven the lower and upper bound

$$1.77 \leq \frac{2^{1-1/d}}{\Gamma(1+1/d)} \leq m_d \leq \frac{2^{1-1/d}\mathrm{e}\Gamma(d+1)^{1/d}}{d} \leq 2.62\,.$$

By the Vershik–Kerov/Logan–Shepp theorem, it is known that $m_2 = 2$, and the bounds give $m_\infty = 2$, but there are no exact expressions for other finite values of the dimension. A numerical estimate for $d = 3$ was established in [Rideout–Wallden 2009], $m_3 \approx 2.296 \pm 0.012$.

For comparison in the following plot of normalised values, we use the order-dimension for d in the case of the uniform probability distribution over causets.

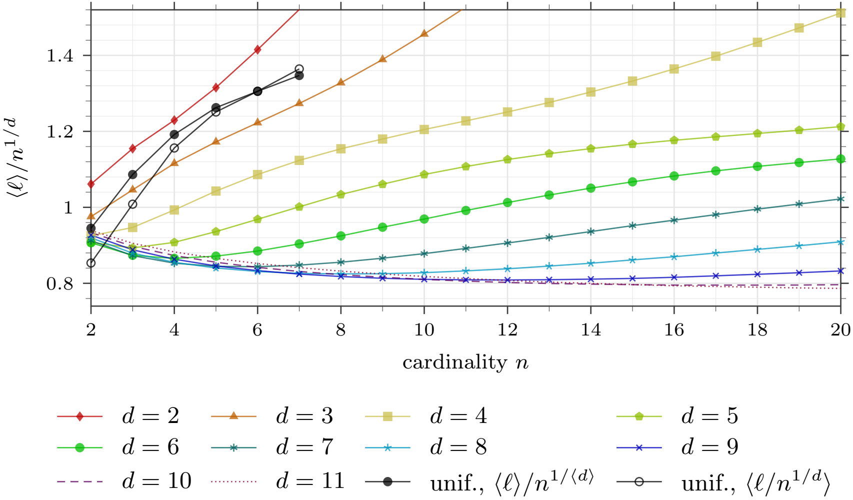


Figure: Normalised expected number of layers over the sprinkling cardinality.

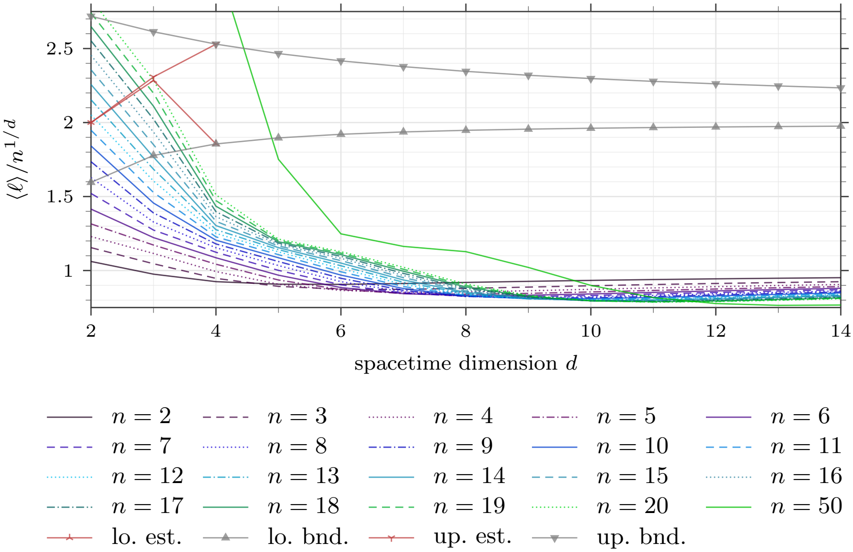


Figure: Normalised expected number of layers over the sprinkling spacetime dimension.

The probability to get a sprinkle isomorphic to C under the condition that $|S| = n = |C|$ is

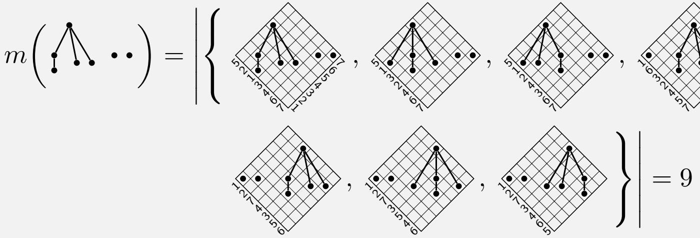
$$\mathbb{P}_{2,n}(C) = n! \sum_p \left(\int_{u_n=0}^{1} \int_{u_{n-1}=0}^{u_n} \cdots \int_{u_1=0}^{u_2} \mathrm{d}u_1 \cdots \mathrm{d}u_{n-1} \mathrm{d}u_n \right) \times \left(\int_{v_{p(n)}=0}^{1} \int_{v_{p(n-1)}=0}^{v_{p(n)}} \cdots \int_{v_{p(1)}=0}^{v_{p(2)}} \mathrm{d}v_{p(1)} \cdots \mathrm{d}v_{p(n-1)} \mathrm{d}v_{p(n)} \right),$$

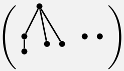
where p runs over all permutations of integers $[n] \coloneqq [1, n]$ that corresponds to the partial order of C. Each of these multi-integrals always yields the same result, $\frac{1}{n!}$ for the u-integrals and $\frac{1}{n!}$ for the v-integrals. So the probability is determined by the number $m(C)$ of permutations (v-coordinate labels) that correspond to the partial order of C,

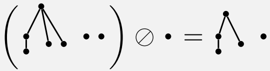
$$\mathbb{P}_{2,n}(C) = \frac{m(C)}{n!} .$$

Example (A 7-element causet with disjoint components)

The multiplicity is easily determined visually, here using the online tool PrOSET editor and the LaTeX-package causets (which also give "faithful" Minkowski space embeddings for 2-orders).



$\mathbb{P}_{2,n}$  $= \frac{9}{7!} = \frac{1}{540}$

Note that this causet has local symmetries: 

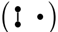
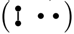
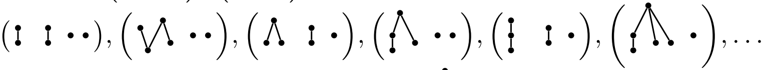
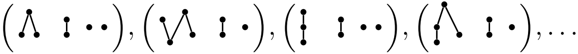
n	most likely causets, $C \in$	$m(C)$
3		2
4		3
5		4
6		6
7		12

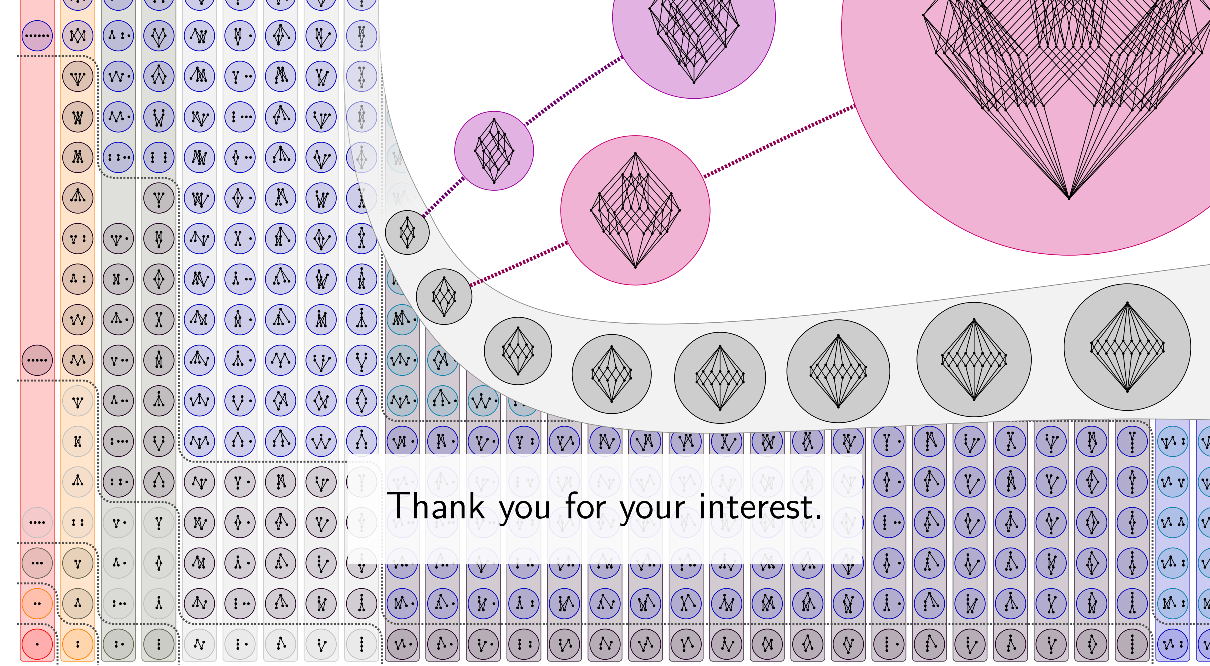
Table: Multiplicities of the most likely causets, where dots indicate that opposite causets have been omitted

•	2	3	4	5	6	7
ratio	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{10}{16}$	$\frac{39}{63}$	$\frac{192}{318}$	$\frac{1179}{2045}$
/%	50.00	60.00	62.50	61.91	60.38	57.65
$\mathbb{E}_{2,n}$	$\frac{1}{2}$	$\frac{3}{6}$	$\frac{14}{24}$	$\frac{70}{120}$	$\frac{430}{720}$	$\frac{3022}{5040}$
/%	50.00	50.00	58.30	58.30	59.72	59.96

	2	3	4	5	6	7
ratio	$\frac{0}{2}$	$\frac{0}{5}$	$\frac{1}{16}$	$\frac{5}{63}$	$\frac{27}{318}$	$\frac{166}{2045}$
/%	0.00	0.00	6.25	7.94	8.49	8.12
$\mathbb{E}_{2,n}$	$\frac{0}{2}$	$\frac{0}{6}$	$\frac{1}{24}$	$\frac{7}{120}$	$\frac{37}{720}$	$\frac{205}{5040}$
/%	0.00	0.00	4.17	5.83	5.14	4.07

Table: Analytic results: the conditional expectation values for sprinkles with at least one singleton symmetry (left) / one prime* 2-chain symmetry (right)

* For the definition of local (prime) symmetries, see "Local symmetries in partially ordered sets" [M. 2024].



Thank you for your interest.